Fuzzy Soft Quasi-Ideals and Bi-Ideals Over a Right Ternary Near-Ring

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ABSTRACT

Right ternary near-rings (RTNR) are generalization of their binary counterpart and fuzzy soft sets are generalization of soft sets which are parameterized family of subsets of a universal set. In this paper fuzzy soft N-subgroups, quasi-ideals and bi-ideals over a right ternary near-ring N are defined. The substructures N-subgroups, quasi-ideals and bi-ideals of an RTNR are characterized in terms of fuzzy soft N-subgroups, quasi-ideals and bi-ideals. The homomorphic and the inverse homomorphic image of fuzzy soft bi-ideals are realized as fuzzy soft bi-ideals and also a fuzzy soft bi-ideal over a regular zero-symmetric RTNR is established as a fuzzy soft RTNR. Left regular zero-symmetric RTNR are characterized in terms of fuzzy soft completely semi-prime left N-subgroups. It is also proved that a fuzzy soft lateral N-subgroup is a fuzzy soft N-subgroup in an intra-regular RTNR.

Keywords- Fuzzy soft set, fuzzy soft ideal, zero-symmetric RTNR, regular RTNR, intra-regular RTNR.

1. INTRODUCTION

Since the introduction of the concept of fuzzy sets by Zadeh [16] in 1965 many notions of mathematics are extended to such sets and various properties of these concepts in the context of fuzzy sets are established. In 1999, Molodtsov [10] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. In 2001 Maji et al [9] expanded soft set theory to fuzzy soft sets and in recent times researches have contributed a lot towards fuzzification of soft set theory in algebraic structures viz., groups, rings, modules and semigroups.

The notion of ternary algebraic system was introduced by Lehmer [8] in 1932: Ternary semigroups [11], ternary semirings [7] are some of the algebraic structures which involve ternary product. To deal with the concept of near-rings using ternary product Warud Nakkhasen and Bundit Pibaljommee [15] have applied the concept of ternary semiring to define left ternary near-rings, ternary subnear-rings and their ideals and investigated some properties of L-fuzzy ternary near subrings in 2012.


In 1987, Chelvam and Ganesan [4] introduced and generalized the notion of quasi-ideals of near-rings to bi-ideals. The regular near-ring was introduced by Beidlemann [3]. The concept of fuzzy R-subgroups of a near-ring R was introduced by S. Abou-Zaid [1], and K.H.Kim and Y.B.Jun [6] investigated further properties of fuzzy right (resp. left) R-subgroups.

The authors [12] introduced fuzzy soft right ternary near-rings, fuzzy soft ideals and studied their basic algebraic properties. In this paper fuzzy soft N-subgroups, quasi-ideals and bi-ideals over an RTNR are defined and their basic characteristics are studied. It is established that every fuzzy soft quasi-ideal is a fuzzy soft bi-ideal and over a regular zero-symmetric RTNR a fuzzy soft bi-ideal is a fuzzy soft right ternary near-ring. Left regular zero-symmetric RTNR are characterized in terms of fuzzy soft completely semi-prime left N-subgroups.

2. PRELIMINARIES

In this section the basic definitions that are necessary for the following sections of this paper are given.

2.1 Definition [11]

Let T be a non-empty set and [ ] be an operation defined from N x N x N to N called a ternary operation. Then(N,[ ]) is a ternary semigroup if for every x,y,z,u,v ∈ N, [[xyz]uv] = [x[yzu]v] = [x[yzu]]v] = [x[yzu]v]v].
2.2 Definition [11]
Let A, B, C be non-empty subsets of a ternary semigroup N. Then \([ABC] = \{[abc] \in N | a \in A, b \in B, c \in C\}\).

2.3 Definition 2.3[12]
Let N be a non-empty set together with a binary operation + and a ternary operation 
\([\cdot]: N \times N \times N \to N\). Then (N, +, [\cdot]) is a right ternary near-ring (a right ternary near ring is written as RTNR) if

(RTNR-1) (N, +) is a group.

(RTNR-2) (N, [\cdot]) is a ternary semigroup.

(RTNR-3) \([a + b, c + d] = [a, c] + [b, d]\) for every a, b, c, d \(\in N\).

Similarly left ternary near-ring and lateral ternary near ring are defined.

2.4 Definition [12]

2.5 Definition [15]
Let N and N’ be any two right ternary near rings. Then a mapping \(h: N \to N’\) is called a right ternary near ring homomorphism if

(i) \(h(x + y) = h(x) + h(y)\),
(ii) \(h([x, y, z]) = [h(x), h(y), h(z)]\) for every \(x, y, z \in N\).

2.6 Definition [12]
Let N be a right ternary near-ring. Let J be a normal subgroup (N, +). Then J is called (i) a right ideal of N if \([JNN] \subseteq J\)

(ii) a left ideal if \([t + t’ + i] \subseteq J\)

(iii) a lateral ideal if \([tt’ + it’] \subseteq J\) for every \(t, t’, t'' \in N\), \(i \in J\).

J is an ideal of N if it is a right, lateral and left ideal of N.

2.7 Definition [16]
If X is a universal set then a fuzzy subset of X is a map \(\mu: X \to [0,1]\) which is denoted by \(\mu = \{(x, \mu(x)) | x \in X\}\).

2.8 Definition [10]
Let U be a universal set. Let A be a subset of a set of parameters E. Then \((f, A)\) is called a soft set over U where \(F: A \to \mathcal{P}(U)\)

and \(\mathcal{P}(U)\) is the set of subsets of U.

2.9 Definition [9]
Let U be a universal set and let A and B be any two non-empty subsets of a set of parameters E. Then \((f, A)\) and \((g, B)\) be any two fuzzy soft sets over U. Then

(i) \((f, A)\) is a fuzzy soft subset of \((g, B)\) i.e., \((f, A) \subseteq (g, B)\) if \(A \subseteq B\) and \(f_a(x) \leq g_b(x)\) for every \(a \in A\).

(ii) The intersection of \((f, A)\) and \((g, B)\) such that \(A \cap B \neq \emptyset\) is defined to be the fuzzy soft set \((h, C)\), where \(C = A \cap B\) and \(h(c) = h_a = f_a \cap g_b\) for all \(c \in C\) and \(h_a(u) = \min\{f_a(u), g_b(u)\} = f_a(u) \land g_b(u) = (f_a \lor g_b)(u)\). The intersection of \((f, A)\) and \((g, B)\) is denoted by \((h, C) = (f, A) \cap (g, B)\).

2.12 Definition [2]
Let X and Y be any two non-empty sets and \(E_1\) and \(E_2\) be their parameter sets. Let \(A \subseteq E_1\), \(a \in A\), and \(t \in \text{Im} f_a\). Let \((f, A)\) and \((g, B)\) be any two non-empty fuzzy soft sets over U and V respectively. Let

\[f_a(x) \leq g_b(x) \text{ for every } x \in U\]
φ: X → Y and ψ: A → B. Then (φ, ψ): (f, A) → (g, B) is called a fuzzy soft function and the image set (φ(f), B) of (f, A) under (φ, ψ) is defined as follows.

For every y ∈ Y and b ∈ B

\[
(φ (f))_b(y) = \begin{cases} 
V_{x \in φ^{-1}(y)}(V_{a \in ψ^{-1}(b) \cap A} f_a(x)), & \text{if } φ^{-1}(y) \neq \emptyset, b \in ψ(A) \\
\text{otherwise} & 
\end{cases}
\]

2.13 Definition [2]

Inverse image of fuzzy soft set (g, B) is defined by \((φ^{-1}(g), ψ^{-1}(B)) \) for every \(a \in ψ^{-1}(B)\) and \(x \in X\).

In the following we give the basic definitions and results of fuzzy soft right ternary near-rings and ideals as given in [18] and [19]

2.14 Definition [12]

A fuzzy soft set \((f, A)\) over \(N\) is a fuzzy soft right ternary near-ring if

(i) \(f_a(x + y) \geq \min \{ f_a(x), f_a(y) \}\),
(ii) \(f_a(-x) \geq f_a(x)\) for every \(a \in A, x,y \in X\) and
(iii) \(f_a([xyz]) \geq \min \{ f_a(x), f_a(y), f_a(z) \}\) for every \(a \in A\) and \(x,y,z \in N\).

2.15 Remark [12]

Conditions (i) and (ii) are normally combined together to get the condition \(f_a(x - y) \geq \min \{ f_a(x), f_a(y) \}\) and if this holds then \((f, A)\) is a fuzzy soft subgroup over \(N\).

2.16 Definition [12]

A fuzzy soft set \((f, A)\) over a right ternary near-ring \(N\) is fuzzy soft ideal over \(N\) if

(i) \(f_a(x - y) \geq \min \{ f_a(x), f_a(y) \}\)
(ii) \(f_a(y + x - y) \geq f_a(x)\)
(iii) \(f_a([xyz]) \geq f_a(x)\)
(iv) \(f_a([x y (z + i)] - [x y z]) \geq f(i)\)
(v) \(f_a([x (y + i) z] - [x y z]) \geq f(i)\) for every \(a \in A\) and \(x,y,z,i \in N\).

A fuzzy soft set \((f, A)\) is called a fuzzy soft right ideal if it satisfies (i), (ii) and (iii); \((f, A)\) is called a fuzzy soft left ideal if it satisfies (i), (ii) and (iv); \((f, A)\) is called a fuzzy soft lateral ideal if it satisfies (i), (ii) and (v).

2.17 Definition [13]

\(N_0 = \{ n \in N | [n 0 0] = 0 \}\) is the zero-symmetric part of \(N\) and if \(N = N_0\) then \(N\) is called a zero-symmetric RTNR.

2.18 Definition [13]

Let \(N\) be an RTNR and \(B, C, D\) be subsets of a parameter set \(E\) of \(N\). If \((g, B), (h, C), (r, D)\) are fuzzy soft sets over \(N\), then; \((g, B) \circ (h, C) = (r, D) = (k, L)\) where \(L = B \cap C \cap D\) and \((g_B \circ h_C \circ r_D)(u) = k_L(u) = \begin{cases} 
\sup \{ g_{[x]} \wedge h_{[y]} \wedge r_{[z]}(u) \} & \text{if } u = [xyz] \\
0 & \text{otherwise} \end{cases}
\)

for every \(u \in L\).

2.19 Definition [13]

If \(N\) is an RTNR then a fuzzy soft ideal \((f, A)\) over \(N\) is a fuzzy soft completely semi-prime ideal if \(f_a([xxx]) \leq f_a(x)\), for all \(a \in A\) and \(x \in N\).
2.20 Lemma [13]
If N is an RTNR then a fuzzy soft ideal \( (f, A) \) over N is a fuzzy soft completely semi-prime iff \( f_a(x^i) = f_a(x) \) for all \( a \in A \) and \( x \in N \).

2.21 Definition [14]
A non-empty subset H of N is called an N-subgroup of N if (i) H is a subgroup of \( \langle R, + \rangle \) (ii) \([NNH] \subseteq H \) (iii) \([NHN] \subseteq H \) (iv) \([HNN] \subseteq H \).

2.22 Definition [14]
If N is an RTNR then N is said to be intra-regular if for each element \( x \in N \), there exists elements \( u, v \in N \) such that \([u \cdot x \cdot v] = x \).

2.23 Definition [14] An element \( x \in N \) is called a right (resp. left) regular if there exists \( u \in N \) such that \([xxu] = x \) (resp. \([uxx] = x \)).

3. FUZZY SOFT N-SUBGROUPS
In this section a fuzzy soft N-subgroup is defined and their basic algebraic properties are studied. Throughout this section N is a zero-symmetric RTNR and E is a parameter set associated with N and A is a subset of E.

3.1 Definition
A fuzzy soft set \((f, A)\) over N is called a fuzzy soft right (resp., lateral, left) N-subgroup over N if

1. \( f_a(x \cdot y) \geq \min \{ f_a(x), f_a(y) \} \)
2. \( f_a([xyz]) \geq f_a(x) \) (resp. \( f_a([xyz]) \geq f_a(y), f_a([xyz]) \geq f_a(z) \)) for all \( x, y, z \in N \) and \( a \in A \).

3.2 Proposition
A fuzzy soft left (resp. lateral) ideal over a zero-symmetric RTNR is a fuzzy soft left (resp. lateral) N-subgroup.

Proof: Let \((f, A)\) be a fuzzy soft left ideal over N. Then for all \( x, y, z \in N \) and \( a \in A \), \( f_a(x \cdot y) \geq \min \{ f_a(x), f_a(y) \} \) and \( f_a([xy(z+i)] - [xy z]) \geq f_a(i) \rightarrow (1) \).

Now \( f_a([xyz]) = f_a([x \cdot y (0+z)] - [x y 0]) \geq f_a(z) \), using (1). Hence \((f, A)\) is a fuzzy soft left N-subgroup over N. Similarly a fuzzy soft lateral ideal over a zero-symmetric RTNR is a fuzzy soft lateral N-subgroup over N.

3.3 Example
Let \( N = \{0, x, y, z\} \). Define \(+\) on N as in Table (i) and the ternary operation \([\ ]\) on N by \([x y z] = (x y) z\) for every \( x, y, z \in N \) where ‘.’ is defined as in Table (ii). Then \((N, +, [\ ]\) is a right ternary near-ring.

\[
\begin{array}{|c|c|c|c|}
\hline
+ & 0 & x & y & z \\
\hline
0 & 0 & x & y & z \\
0 & x & 0 & z & y \\
0 & y & z & 0 & x \\
0 & z & y & x & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|}
\hline
\cdot & 0 & x & y & z \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & x \\
0 & y & 0 & x & y \\
0 & z & 0 & x & y \\
\hline
\end{array}
\]

Table (i) Table (ii)

Let \( A = \{0, x\} \) and \( f : A \rightarrow 1^N \) be defined by \( f_0(0) = 1 = f_0(x) \) and \( f_0(z) = 0.6 \) for every \( a \in A \). Then \((f, A)\) is a fuzzy soft N-subgroup over N.

On the other hand, if \( B = \{0, y\} \) and \( g : B \rightarrow 1^N \) is defined by \( g_b(0) = 1 = g_0(y) \) and \( g_b(x) = 0.8 = g_b(z) \) for every \( b \in B \) then \((g, B)\) is only a left N-subgroup but not a fuzzy soft right or left N-subgroup over N as \( g_b([xyz]) = g_b(x) = 0.8 \geq 1 = g_b(y) \) and \( g_b([zyx]) \neq g_b(x) = 0.8 \geq 1 = g_b(y) \).

Also if \( C = \{0, z\} \) and \( h : C \rightarrow 1^N \) is defined by \( h_c(0) = 1 = h_c(z) \) and \( h_c(y) = 0.7 = h_c(x) \) for every \( c \in C \) then \((g, B)\) is not a fuzzy soft right or lateral or left N-subgroup over N as \( h_c([zxz]) = h_c(x) = 0.7 \geq 1 = h_c(z) \) and \( h_c([xzx]) \neq h_c(x) = 0.7 \geq 1 = h_c(z) \).
### 3.4 Lemma

If \((f, A)\) is a fuzzy soft set over \(N\) such that for all \(a \in A\) and \(x,y,z \in N\) \(f_a(x, z) \geq \min\{ f_a(x), f_a(y) \}\) then

(i) \(f_a([xyz]) \geq \min\{ f_a(x), f_a(y), f_a(z) \}\) i.e., \((f, A)\) is a fuzzy soft RTNR iff \((f, A) \not{\in} (f, A) \subseteq (f, A)\).

(ii) \(f_a([xyz]) \geq f_a(x)\) i.e., \((f, A)\) is a fuzzy soft left \(N\)-subgroup iff \((f, A) \not{\in} (1, E) \not{\in} (1, E) \subseteq (f, A)\).

(iii) \(f_a([xyz]) \geq f_a(z)\) i.e., \((f, A)\) is a fuzzy soft right \(N\)-subgroup iff \((1, E) \not{\in} (f, A) \subseteq (f, A)\).

Proof: Let \((f, A)\) be a fuzzy soft set over \(N\) such that

\(f_a(x, y) \geq \min\{ f_a(x), f_a(y) \}\).

(i) Suppose \(f_a([xyz]) \geq \min\{ f_a(x), f_a(y), f_a(z) \}\) i.e., \((f, A)\) is a fuzzy soft RTNR. Let \((f, A) \not{\in} (f, A) = (h, A)\). If \(u \neq 0\) and \(u = [xyz]\) then \(h_a(u) = \sup_{u=[xyz]}(f_a(x) \land f_a(y) \land f_a(z))\Rightarrow h_a(u) = f_a([xyz])= f_a(u)\), for all \(a \in A\). Hence \((f, A) \not{\in} (f, A) \subseteq (f, A)\). Conversely if \((f, A) \not{\in} (f, A) \subseteq (f, A)\) i.e., \((h, A) \subseteq (f, A)\) then

\[h_a(u) = \sup_{u=[xyz]}(f_a(x) \land f_a(y) \land f_a(z)) \leq f_a(u)\]. Thus \(\min\{ f_a(x), f_a(y), f_a(z) \}\)

\[\leq f_a(u) = f_a([xyz])\] for all \(a \in A\).

(ii) Let \(f_a([xyz]) \geq f_a(x) \Rightarrow (f, A)\) be a fuzzy soft left \(N\)-subgroup. Let \((f, A) \not{\in} (1, E) \not{\in} (1, E) = (h, A)\). Then \(h_a(u) = \sup_{u=[xyz]}(f_a(x) \land 1_{f_a(z)} \land 1_{f_a(z)})\Rightarrow h_a(u) = f_a([xyz])= f_a(u)\), by (2) for all \(a \in A\). Hence \((f, A) \not{\in} (1, E) \not{\in} (1, E) \subseteq (f, A)\). Conversely if \((f, A) \not{\in} (1, E) \not{\in} (f, A)\) i.e., \((h, A) \subseteq (f, A)\) then \(h_a(u) = \sup_{u=[xyz]}(f_a(x) \land 1_{f_a(z)} \land 1_{f_a(z)}) \leq f_a(u) \Rightarrow f_a(x) \leq \sup_{u=[xyz]}(f_a(x)) \leq f_a(u) = f_a([xyz])\) for all \(a \in A\).

Similarly (iii) and (iv) can be established.

### 3.5 Remark

Since in a zero-symmetric RTNR a fuzzy soft ideal is an \(N\)-subgroup the following Lemma 3.6, Theorem 3.7, Theorem 3.8, Proposition 3.9 and Proposition 3.10 can be proved on similar lines for \(N\)-subgroups as that for fuzzy soft ideals given in [12].

### 3.6 Lemma

If \((f, A)\) is a fuzzy soft right (resp., lateral, left) \(N\)-subgroup over \(N\) then

(i) \(f_a(x) = f_a(x)\), (ii) \(f_a(0) \geq f_a(x)\), (iii) if \(f_a(x-y) = f_a(0)\), then \(f_a(x) = f_a(y)\), for every \(a \in A\) and \(x, y \in N\).

### 3.7 Theorem

If \((f, A)\) is a fuzzy soft set over \(N\) and if for \(a \in A\) \(S = \{x \in N | f_a(x) = f_a(0)\}\) then \(S\) is an \(N\)-subgroup of \(N\) if \((f, A)\) is a fuzzy soft \(N\)-subgroup over \(N\).

### 3.8 Theorem

(i) A non-empty subset \(L\) of \(N\) is an \(N\)-subgroup of \(N\) iff \((f, A)\) is a fuzzy soft \(N\)-subgroup over \(N\), where \(f: A \rightarrow \text{I}^N\) is defined by

\[f_a(x) = \begin{cases} r & \text{if } x \in L \\ t & \text{if } x \in N - L \end{cases}\] where \(r > t\) for every \(a \in A\) and \(r \in [0,1]\).

(ii) In particular, a non-empty subset \(H\) of \(N\) is an \(N\)-subgroup of \(N\) iff the characteristic function \((\Psi_h, E)\) is a fuzzy soft \(N\)-subgroup over \(N\), where \(\Psi_H : E \rightarrow \text{I}^N\) is defined by

\[(\Psi_h)(x) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{if } x \in \overline{H} \end{cases}, \text{for every } e \in E.\]

### 3.9 Proposition

The intersection of two non-empty fuzzy soft \(N\)-subgroups over \(N\) is a fuzzy soft \(N\)-subgroup over \(N\).
3.10 Proposition

(f, A) is a fuzzy soft N-subgroup over N iff for each \( f_a \), each non-empty level subset (\( f_a \)), \( a \in \text{Im} f_a \) is an N-subgroup of N.

3.11 Proposition

Let (f, A) be a fuzzy soft right N-subgroup over N. Let \( \varphi: N \rightarrow N \) be a homomorphism and define \( f^\ast_a(x) = f_a(\varphi(x)) \) for every \( x \in X \) and \( a \in A \). Then \((f^\ast, A)\) is also a fuzzy soft right N-subgroup over N.

Proof: Consider, \( f^\ast_a(x-y) = f_a(\varphi(x-y)) = f_a(\varphi(x) - \varphi(y)) \geq \min \{ f_a(\varphi(x)), f_a(\varphi(y)) \} = \min \{ f_a(0) + f_a(0), f_a(0) + 1 \} = 1 \) for all \( a \). Hence \((f^\ast, A)\) is a fuzzy soft right N-subgroup over N.

3.12 Note

The case for fuzzy soft lateral and left N-subgroups can be similarly proved.

3.13 Definition

A fuzzy soft N-subgroup \((f, A)\) over N is said to be normal if \( f_a(0) = 1 \) for every \( a \in A \) and \( x \in N \).

3.14 Proposition

Let (f, A) be a fuzzy soft right (resp. lateral, left) N-subgroup over N and \((f^\ast, A)\) be a fuzzy soft set over N where \( f^\ast_a(x) = f_a(x) + 1 - f_a(0) \) for every a in A and x in N. Then (f, A) is a normal fuzzy soft right (resp. lateral, left) N-subgroup over N and \((f^\ast, A) \subseteq (f, A)\).

Proof: Consider \( f^\ast_a(x-y) = f_a(x-y) + 1 - f_a(0) \geq \min \{ f_a(x), f_a(y) \} + 1 - f_a(0) \geq \min \{ f_a(x) + 1 - f_a(0), f_a(y) + 1 - f_a(0) \} = \min \{ f_a(x), f_a(y) \} \) for all \( a \). Hence \((f^\ast, A)\) is a fuzzy soft right N-subgroup over N. Also, \( f_a^\ast(0) = 1 \) and hence \((f^\ast, A)\) is a normal fuzzy soft right N-subgroup over N. Since \( f_a(x) \leq f_a^\ast(x), (f^\ast, A) \subseteq (f, A) \) for all \( a \), thus completing the proof.

The case for fuzzy soft lateral and left N-subgroups can be similarly proved.

3.15 Theorem

Let (f, A) be a fuzzy soft right (resp. lateral, left) N-subgroup over N. Let \((f^\ast, A)\) be a fuzzy soft set over N where \( f^\ast_a(x) = f_a(x)/f_a(0) \). Then \((f^\ast, A)\) is a normal fuzzy soft right N-subgroup (resp. lateral, left) over N and also \((f^\ast, A) = (f, A)\) iff (f, A) is a normal fuzzy soft N-subgroup.

Proof: \( f^\ast_a(x-y) = f_a(x-y) + 1 - f_a(0) \geq f_a(x)/f_a(0) + f_a(y)/f_a(0) \geq \min \{ f_a(x)/f_a(0), f_a(y)/f_a(0) \} = \min \{ f^\ast_a(x), f^\ast_a(y) \} \) for all \( a \). Hence \((f^\ast, A)\) is a normal fuzzy soft right N-subgroup over N. The case for fuzzy soft lateral and left N-subgroups can be similarly proved. Obviously \((f^\ast, A) = (f, A)\) if (f, A) is a normal fuzzy soft N-subgroup. Conversely, if \( f^\ast_a(x) = f_a(x) \) then \( f_a(0) = 1 \) which implies that (f, A) is a normal fuzzy soft N-subgroup.

4. Fuzzy Soft Quasi-ideals

In this section fuzzy soft quasi-ideals are defined and its basic algebraic properties are studied.

4.1 Definition

Let N be a zero-symmetric RTNR and A be a subset of set of parameters E of N. Then a fuzzy soft set \((f, A)\) over N is called a fuzzy soft quasi-ideal over N if

(i) \( f_a(x-y) \geq \min \{ f_a(x), f_a(y) \} \).

(ii) \( (f_a \circ 1_e) \cap (1_e \circ f_a \circ 1_e + 1_e \circ f_a \circ 1_e) \subseteq f_a \).

Equivalently,

\[
\min \{ (f_a \circ 1_e)(u), (1_e \circ f_a \circ 1_e + 1_e \circ f_a \circ 1_e)(u), (1_e \circ 1_e \circ f_a)(u) \} \leq f_a(u)
\]

where

\[
(1_e \circ f_a \circ 1_e + 1_e \circ 1_e \circ f_a \circ 1_e)(u)
\]
Thus \((f, A)\) is a fuzzy soft quasi-ideal over \(N\). On the other hand if \(B = \{0, z\}\) and \(g : B \rightarrow I^N\) is defined by \(g_b(0) = 1 \Leftrightarrow g_b(1) = 0.7 = g_b(z)\) for every \(b \in B\) then \((g, B)\) is not a fuzzy soft quasi-ideal over \(N\).

For if \(u = x = [zxz]\) then \((g_b \circ 1 \circ 1_e)(u) = 1\) and \((1_e \circ 1_e \circ g_b)(u) = 1\).

Now as \(u = [zxz] + [x\{0\}y]x\), \(u = [xzx] + [x\{0\}y]x\), etc.

\((1_e \circ g_b \circ 1_e \circ 1_e \circ g_b \circ 1_e)(u) = \sup\{g_b(x), g_b(z) \land g_b(0), \ldots\}\)

Hence \((1_e \circ g_b \circ 1_e \circ 1_e \circ g_b \circ 1_e)(u) = 1\).

Thus \(\min\{(g_b \circ 1 \circ 1_e)(u), (1_e \circ g_b \circ 1_e \circ 1_e \circ g_b \circ 1_e)(u), (1_e \circ 1_e \circ g_b(0)) = 1\) and \(g_b(u) = 0.7\). Hence \((g, B)\) is not a fuzzy soft quasi-ideal over \(N\).

### 4.3 Lemma

If \(N\) is a zero-symmetric RTNR then

(i) Every fuzzy soft right (resp. lateral, left) \(N\)-subgroup over \(N\) is a fuzzy soft quasi-ideal over \(N\).

(ii) Every fuzzy soft right (resp. lateral, left) ideal over \(N\) is a fuzzy soft quasi-ideal over \(N\).

**Proof:** (i) Let \((f, A)\) be a fuzzy soft right (resp. lateral, left) \(N\)-subgroup over \(N\). Then \(f_x \geq \min\{f_x, f_y\} \) and \(f_x (xy) \geq f_y (x) \) for every \(x, y \in N\). Now let \(u \in N\). If \(u = 0\) then \(u = [0xy] = [0y] \) so that \((f_x \circ 1_e \circ 1_e)(u) = f_x(u)\) by Lemma 3.6(ii). Also \((1_e \circ f_x \circ 1_e \circ 1_e \circ 1_e)(u) = f_x(u) \) (as \(N\) is a RTNR over \(N\). Hence if \(u = 0\) then \(min\{(f_x \circ 1_e \circ 1_e)(u), (1_e \circ f_x \circ 1_e \circ 1_e)(u), (1_e \circ 1_e \circ f_x)(u)\} = f_x(u)\) if \(u \neq 0\), let \(u = [y]x\)

\(f_x(u) = \sup\{f_x(y) \land 1 \land 1\} = \sup\{f_x(x) \circ f_x(y) \circ 1_e \circ 1_e \circ f_x\}(u)\). Thus \((f, A)\) is a fuzzy soft right \(N\)-subgroup over \(N\). Hence \(\min\{(f_x \circ 1_e \circ 1_e)(u), (1_e \circ f_x \circ 1_e \circ 1_e)(u), (1_e \circ 1_e \circ f_x)(u)\} \subseteq (f_x \circ 1_e \circ 1_e)(u) \leq f_x(u)\).

(ii) Since in a zero-symmetric RTNR a fuzzy soft ideal is an \(N\)-subgroup by (i) the proof follows.

### 4.4 Remark

The converse of the above lemma is not true. For if \(N\) is as in Example 3.3 and \(A = \{0, y\}\) and \(f : A \rightarrow I^N\) is defined by \(f_0(0) = 1 \Leftrightarrow f_0(1) = 0.9\) for every \(a \in A\) then \((f, A)\) is a fuzzy soft quasi-ideal over \(N\) but \(f_0([yxy]) = f_0(x) = 0.9 \geq 1 = f_0(y)\) which implies that \((f, A)\) is not a fuzzy soft right \(N\)-subgroup.

### 4.5 Lemma

Every fuzzy soft quasi-ideal over \(N\) is a fuzzy soft RTNR over \(N\).

**Proof:** Let \((f, A)\) be a fuzzy soft quasi-ideal over \(N\). Then \(f_x \geq \min\{f_x, f_y\}\) for every \(a \in A\) and \(x, y \in N\). Let \(u = [xy]\). Then \((f_x \circ 1_e \circ 1_e)(u) = \sup\{f_x(u) \land 1 \land 1\} = \sup\{f_x(u) \circ 1_e \circ 1_e \circ f_x\}(u)\). Similarly \((1_e \circ f_x \circ 1_e \circ 1_e \circ 1_e)(u) \geq f_x(u)\).

Now \(\min\{(f_x \circ 1_e \circ 1_e)(u), (1_e \circ f_x \circ 1_e \circ 1_e)(u), (1_e \circ 1_e \circ f_x)(u)\} \leq \min\{f_x(u), f_y(u), f_z(u)\} \). Thus \((f, A)\) is a fuzzy soft RTNR over \(N\).

### 4.6 Proposition

The intersection of two fuzzy soft quasi-ideals over \(N\) is a fuzzy soft quasi-ideal over \(N\).

**Proof:** Let \((f, A)\) and \((g, B)\) be any two fuzzy soft sets. Let \((h, C) = (f, A) \cap (g, B)\) where \(C = A \cap B\) and \(h(u) = \min\{f(u), g(u)\}\). Therefore \(h_{x-y} = \min\{f_{x-y}, g_{x-y}\} \geq \min\{f_{x-y}, f_{y-x}\}, \min\{f_{x-y}, g_{x-y}\}\) for every \(a \in A\) and \(u \in N\).
Now if u = 0, then it is obvious that \( \{h_1, h_2, h_3\} \). Let x, y ∈ \( \mathbb{N} \). Then k = \( \min \{\min \{\min \{f_1(x), f_2(y)\} \}, \min \{\min \{f_2(x), f_3(y)\} \}\}. 

Thus \( \min \{h_1, h_2, h_3\} \leq h\). Hence (h, C) is a fuzzy soft quasi-ideal over N.

### 4.7 Corollary

The arbitrary intersection of fuzzy soft quasi-ideals over N is a fuzzy soft quasi-ideal over N.

### 4.8 Lemma

If \((f, A), (g, B), (h, C)\) are fuzzy soft right, lateral and left N-subgroups respectively then \((f, A) \cap (g, B) \cap (h, C) = (K, D)\) is a fuzzy soft quasi-ideal over N.

### 4.9 Theorem

If \((f, A), (g, B), (h, C)\) are fuzzy soft right, lateral and left N-subgroups respectively then \((f, A) \cap (g, B) \cap (h, C) = (K, D)\) is a fuzzy soft quasi-ideal over N.

### 4.10 Corollary

If \((f, A), (g, B)\) are fuzzy soft right and left N-subgroups respectively then \((f, A) \cap (g, B) = (K, D)\).

### 4.11 Remark

Theorem 4.9 and Corollary 4.10 hold good for fuzzy soft right, lateral and left ideals also.

### 4.12 Theorem

A fuzzy soft set \((f, A)\) is a fuzzy soft quasi-ideal over N iff for each \(f_i\), each non-empty level subset \((f_i)_k\), \(k \in \text{Im } f_i\) is a quasi-ideal of N.

### Proof

Let \((f, A)\) be a fuzzy soft quasi-ideal over N. Let \(t \in \text{Im } f_i\) be such that \((f_i)_k \neq \emptyset\). Let \(x, y \in (f_i)_k\). Since \(f_i(x, y) \geq \min \{f_i(x), f_i(y)\} \geq t, x - y \in (f_i)_k\).

Let \(u \in \{f_i(x), f_i(y)\}\). Consider \(u = [xyz] = [x\ y\ z]\) where \(x, y, z \in (f_i)_k\). This implies that \(f_i(x) \geq t, f_i(y) \geq t, f_i(z) \geq t\). Since \(f_i([xyz]) = f_i(u) \geq \min \{f_i(x), f_i(y)\} \geq \min \{f_i(x), f_i(y)\} \geq t\). Hence \(u \in (f_i)_k\) and \((f_i)_k\) is a quasi-ideal of N.

Conversely, let \(x, y \in N\). Suppose \(f_i(x, y) < \min \{f_i(x), f_i(y)\}\). Let \(s = \min \{f_i(x), f_i(y)\}\). Then as \(x \in (f_i)_k\), \(y \in (f_i)_k\), \(x - y \in (f_i)_k\) and hence \(f_i(x, y) < \min \{f_i(x), f_i(y)\}\), a contradiction to the assumption.

Hence \(f_i(x, y) \geq \min \{f_i(x), f_i(y)\}\). Let \(u \in N\). Suppose \(f_i(u) < \min \{f_i(x), f_i(y)\}\). Then \(f_i(u) < \min \{f_i(x), f_i(y)\}\).

Then \(f_i(x, y) \geq r = \{f_i(x), f_i(y)\} \geq t\). Hence \(f_i(x, y) \geq \min \{f_i(x), f_i(y)\}\).
Similarly $f_2(x) \geq r, f_2(z) \geq r.$ Thus $x,y,z \in (f_2).$ Since $(f_2)$ is also an RTSNR, $u \in (f_2) \Rightarrow f_2(u) = f_2([xyz]) \geq r \Rightarrow f_2(u) \geq r$ which contradicts (3). Hence min $\{ f_2(1,0,1,0), (1,0,1,0,1) \} \leq f_2(u).$ Thus $(f_2, A)$ is a fuzzy soft quasi-ideal over $N.$

### 4.13 Theorem

A non-empty subset $L$ of $N$ is a quasi-ideal of $N$ iff $(f, A)$ is a fuzzy soft quasi-ideal over $N,$ where $f : A \rightarrow I^N$ is defined by

$$f_2(x) = \begin{cases} r & \text{if } x \in L \\ t & \text{if } x \notin L \end{cases}$$

where $r > t.$ for every $a \in A$ and $r \in [0,1].$

(ii) In particular, a non-empty subset $H$ of $N$ is a quasi-ideal of $N$ iff the characteristic function $(\Psi_H, E)$ is a fuzzy soft quasi-ideal over $N,$ where $\Psi_H : E \rightarrow I^N$ is defined by

$$\Psi_H(x) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

for every $x \in E.$

Proof: Let $L$ be a quasi-ideal of $N.$ Then $L$ is an additive subgroup of $N.$ If $x,y \in L$ then either (i) $x,y \in L$ or (ii) $x \in L, y \notin L$ or (iii) $x,y \notin L.$ Let $(f, A)$ be a fuzzy soft set over $N$ where $f : A \rightarrow I^N$ is defined by

$$f_2(x) = \begin{cases} r & \text{if } x \in L \\ t & \text{if } x \notin L \end{cases}$$

where $r > t.$ for every $a \in A$ and $r \in [0,1].$

Case (i): As $x,y \in L$ and hence $f_2(x,y) = r = \min \{ f_2(x), f_2(y) \}.$

Case (ii): Since $y \notin L,$ $x,y \notin L \Rightarrow f_2(x,y) = t = \min \{ f_2(x), f_2(y) \}.$ Let $u \in N.$ Then either $u \in [LN][NN] \cup [NL]N[N] \cup [NN]N$ or $u \notin [LN][NN] \cup [NL][NL] \cup [NN][NN].$ Consider the case when $u \in [LN][NN] \cup [NL][NL] \cup [NN][NN].$ Let an element $u = [xyz] = [xyz] + [p[qqms]v], where x,y,z \in L.$ Since $x,y,z \in N.$

Case (iii): As $x \notin L, x,y \notin L \Rightarrow f_2(x) = t = \min \{ f_2(x), f_2(y) \}.$

Now $\{ f_2(1,0,1,0) \} = r.$ Similarly $L$ is a quasi-ideal of $N.$ This contradiction leads to conclude that $u \in L.$ Since $L$ is a quasi-ideal of $N.$ Now consider the case when $u \notin [LN][NN] \cup [NL][NL] \cup [NN][NN].$ Let $u = [xyz] = [xyz] + [p[qqms]v], where x,y,z \in N.$

Since $x,y,z \in L,$

$$f_2(x) = t, f_2(x) = t, f_2(x) = t.$$

Now $\{ f_2(1,0,1,0) \} = r.$ Similarly $L$ is a quasi-ideal over $N.$ Conversely, let $(f, A)$ be a fuzzy soft quasi-ideal over $N,$ where $f : A \rightarrow I^N$ is defined as in the hypothesis. Now let $x,y \in E.$ Then $f_2(x) = t, f_2(y) = r.$ This implies that $\min \{ f_2(x), f_2(y) \} = r.$ Since $f_2(x,y) \geq r.$

### 5. Fuzzy Soft Bi-ideals

In this section, fuzzy soft bi-ideals over a zero-symmetric RTNR are defined. It is established that every fuzzy soft quasi-ideal is a fuzzy soft bi-ideal and over a regular zero-symmetric RTNR a fuzzy soft bi-ideal is a fuzzy soft right ternary near-ring. Left regular RTNRs are characterized in terms of fuzzy soft completely semi-prime left $N$-subgroups.

#### 5.1 Definition

If $N$ is a zero-symmetric RTNR then a fuzzy soft set $(f, A)$ over $N$ is called a fuzzy soft bi-ideal of $N$ if

(i) $f_2(x,y) \geq \min \{ f_2(x), f_2(y) \}$

(ii) $f_2([xyz]) \geq \min \{ f_2(x), f_2(y), f_2(z) \}$ for all $a \in A$ and $x,y,z \in N.$
5.2 Example

(i) Let \( N = \{0, x, y, z\} \) and let + be defined as in Table (iii). Let the ternary operation \([\ ]\) on \( N \) be defined by \([xyz] = (x.y).z\) where ‘.’ is defined as in Table (iv). Then \( N \) is a regular zero-symmetric RTNR.

(ii) Let \( N \) be as in Example 3.3. Then \( N \) is not a regular zero-symmetric RTNR as \( x \neq [0x0] \), \( x \neq [xxx] \), \( x \neq [xyx] \) and \( x \neq [zxz] \). Now let \( E = N \), \( B = \{0, x\} \) and \( g : B \to I^N \) is defined by \( g_0(0) = g_0 = g_0(z) \) and \( g_0(y) = g_0(z) = 0.6 \) for every \( b \in B \) then \( g_{(B)} \) is not a fuzzy soft bi-ideal over \( N \) as \( g_0([z][zxz][z]) = g_0(z) = 0, 0.6 \neq 1 = \min \{g_0(z), g_0(z), g_0(z)\} \).

5.3 Proposition

If \( N \) is a zero-symmetric RTNR and \((f, A)\) is a fuzzy soft set over \( N \) such that \( f_a(x) \geq \min \{f_a(x), f_a(y)\}\) then \((f, A)\) is a fuzzy soft bi-ideal over \( N \) iff \((f, A) \aleph (1, E) \aleph (f, A) \aleph (1, E) \aleph (f, A) \subseteq (f, A) \).

Proof: Let \((f, A)\) be a fuzzy soft set over \( N \) such that \( f_a(x) \geq \min \{f_a(x), f_a(y)\}\). Let \((h, A) = (f, A) \aleph (1, E) \aleph (f, A) \aleph (1, E) \aleph (f, A) \subseteq (f, A) \).

5.4 Lemma

Every fuzzy soft quasi-ideal is a fuzzy soft bi-ideal over a zero-symmetric right ternary near-ring \( N \).

Proof: Let \((f, A)\) be a fuzzy soft quasi-ideal over \( N \). Then \( f_a(x.y) \geq \min \{f_a(x), f_a(y)\}\). Let \( \mu = N \) and \( u = [x [myn]z] \). Then as \((f, A)\) be a fuzzy soft quasi-ideal over \( N \), \( f_a(u) \geq \min \{f_a(x), f_a(y), f_a(z)\}\) for all \( a \in A \) and \( x, y, z \in N \). Let \( t = [x][yv][z] \). Then \( h_a(t) = \sup \{f_a(x) \land f_a(y) \land f_a(z)\} \) for all \( [x][yv][z] \). Hence \((f, A)\) is a fuzzy soft bi-ideal over \( N \).

5.5 Lemma

Let \( N \) be a zero-symmetric RTNR. Then

(i) Every fuzzy soft right (resp. lateral, left) \( N \)-subgroup over \( N \) is a fuzzy soft bi-ideal over \( N \).

(ii) Every fuzzy soft right (resp. lateral, left) ideal over \( N \) is a fuzzy soft bi-ideal over \( N \).

Proof: (i) Since Every fuzzy soft right (resp. lateral, left) \( N \)-subgroup over \( N \) is a fuzzy soft quasi-ideal over \( N \) by the above lemma the proof follows.

(ii) By Proposition 3.2 every fuzzy soft right (resp. lateral, left) ideal over \( N \) is a fuzzy soft \( N \)-subgroup over \( N \) and hence by (i) the proof follows.

The proofs of the following Theorem 5.6 and Theorem 5.7 are similar to that of Theorem 4.12 and Theorem 4.13 respectively.
5.6 Theorem
A fuzzy soft set \((f, A)\) is a fuzzy soft bi-ideal over a zero-symmetric right ternary near-ring \(N\) iff for each \(f_s\), each non-empty level subset \((f_s)_a\), \(t \in \text{Im} f_s\) is a bi-ideal of \(N\).

5.7 Theorem
A non-empty subset \(L\) of a zero-symmetric right ternary near-ring \(N\) is a bi-ideal of \(N\) iff \((f, A)\) is a fuzzy soft bi-ideal over \(N\), where \(f : A \rightarrow \text{I}^N\) is defined by
\[
f_s(x) = \begin{cases} r & \text{if } x \in L \\ t & \text{if } x \in N - L \end{cases}
\]
where \(r > t\). for every \(a \in A\) and \(r \in [0,1]\).

(ii) In particular, a non-empty subset \(H\) of \(N\) is a bi-ideal of \(N\) iff the characteristic function \((\Psi, E)\) is a fuzzy soft bi-ideal over \(N\), where \(\Psi : E \rightarrow \text{I}^N\) is defined by
\[
(\Psi_h)(x) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{if } x \in N - H \end{cases}
\]
for every \(e \in E\).

5.8 Lemma
Let \(\phi : N \rightarrow M\) be an onto zero-symmetric right ternary near-ring homomorphism. Let \(E_1\) and \(E_2\) be parameter sets for \(N\) and \(M\) respectively. Let \(\psi : A \rightarrow B\) where \(A \subseteq E_1\), \(B \subseteq E_2\), be a one-to-one mapping such that \(\psi(a) = b\) where \(a \in A\) and \(b \in B\) . Then the image of \((f, A)\) under \((\phi, \psi)\) is a fuzzy soft bi-ideal over \(M\) if \((f, A)\) a fuzzy soft bi-ideal over \(N\).

Proof: Let \(u, v \in M\). Since \(\phi\) is onto there exists \(x, y\) respectively in \(N\) such that \(\phi(x) = u, \phi(y) = v\). Also, \(\phi(x - y) = \phi(x) - \phi(y) = u - v\) and \(\phi(x) - \phi(y) = u - v\). Hence \(\phi([xysz]) = \phi([x] \phi([y])) = \phi([xysz])\) is a fuzzy soft bi-ideal over \(M\).

Hence \(\phi([xysz]) = \phi([x] \phi([y])) = \phi([xysz])\) is a fuzzy soft bi-ideal over \(M\).

5.9 Lemma
Let \(\phi : N \rightarrow M\) be an onto right ternary near-ring homomorphism. Let \(E_1\) and \(E_2\) be parameter sets for \(N\) and \(M\) respectively. Let \(\psi : A \rightarrow B\) where \(A \subseteq E_1\), \(B \subseteq E_2\), be a fuzzy soft set over \(M\). Then \((\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))\) is a fuzzy soft bi-ideal over \(N\).

Proof: Let \(\phi^{-1}(g) = h, \psi^{-1}(B) = C\) where \(h : C \rightarrow N\) and \(h_c : N \rightarrow I\) and also \(h_c(x) = g_{\psi(c)}(\phi(x))\) and hence
\[
h_c(x - y) = g_{\psi(c)}(\phi(x) - \phi(y)) \geq g_{\psi(c)}(\phi(x)) - g_{\psi(c)}(\phi(y))
\]
\[
= \min\{g_{\psi(c)}(\phi(x)), g_{\psi(c)}(\phi(y))\}
\]
It is also seen that \(h([xysz]) = g_{\psi(c)}(\phi([xysz])) = g_{\psi(c)}(\phi([x] \phi([y])) = \min\{g_{\psi(c)}(\phi(x)), g_{\psi(c)}(\phi(y))\}
\]
\[
= \min\{h_c(x), h_c(y), h_c(z)\}.
\]
Hence \((\phi, \psi)^{-1}(g, B)\) is a fuzzy soft bi-ideal over \(N\).

5.10 Lemma
The intersection of two fuzzy soft bi-ideals over zero-symmetric RTNR is a fuzzy soft bi-ideal.

Proof: Let \(N\) be a zero-symmetric right ternary near-ring and \(A, B\) be two subsets of a parameter set \(E\) with \(A \cap B \neq \emptyset\). Let \((f, A)\) and \((g, B)\) be any two non-empty fuzzy soft bi-ideals over \(N\). To prove that \((f, A) \cap (g, B) = (h, C)\) is also a fuzzy soft bi-ideal over \(N\). Consider \(h(x-y) = \min\{f(x-y), g(x-y)\} \geq \min\{f(x), f(y), g(x), g(y)\} = \min\{f(x), g(x), f(y), g(y)\} = \min\{h(x), h(y)\}.\)

Also, \(h([xysz]) = g_{\psi(c)}(\phi([xysz])) = \min\{g_{\psi(c)}(\phi(x)), g_{\psi(c)}(\phi(y)), g_{\psi(c)}(\phi(z))\} = \min\{h_c(x), h_c(y), h_c(z)\} = \min\{h(x), h(y), h(z)\}\) for every \(c \in C\) and \(x, y, z \in N\). Hence \((h, C)\) is also a fuzzy soft bi-ideal over \(N\).

5.11 Theorem
A fuzzy soft bi-ideal over a regular zero-symmetric RTNR is a fuzzy soft RTNR.
Proof: Let $N$ be a regular zero-symmetric RTNR and $A$ be a subset of a parameter set $E$ of $N$. If $(f, A)$ is a fuzzy soft bi-ideal over $N$ then $f_1(x,y) \geq \min\{f_2(x), f_3(y)\}$ and $(f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (f, A)$. Since $(f, A) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A)$ is a fuzzy soft RTNR.

5.12 Theorem

Let $N$ be a regular zero-symmetric RTNR and let $(f, A), (g, B)$ and $(h, C)$ be fuzzy soft right, lateral and left $N$-subgroups over $N$ respectively. Then $(f, A) \supseteq (g, B) \supseteq (h, C) \supseteq (f, A) \supseteq (g, B) \supseteq (h, C) \supseteq (f, A) \supseteq (g, B) \supseteq (h, C) \supseteq (f, A)$.

Proof: Let $(f, A) \supseteq (g, B) \supseteq (h, C) = (k, D)$. Then by Theorem 4.9, $(k, D) \subseteq (f, A) \supseteq (g, B) \supseteq (h, C)$ where $D = A \cap B \cap C$ and $u = [u]y_1y_2 = [u]y_1[y_2]u$. Then $f_1(x, y) = \min\{f_2(x), g_3(y), h_4(u)\} = f_1(x, y)$. Thus $(f, A) \supseteq (g, B) \supseteq (h, C) \supseteq (f, A) \supseteq (g, B) \supseteq (h, C) \supseteq (f, A)$.

Hence by Corollary 4.8, $(f, A) \supseteq (g, B) \supseteq (h, C)$ is a fuzzy soft bi-ideal over $N$.

5.13 Corollary

Let $N$ be a regular zero-symmetric RTNR and let $(f, A), (g, B)$ and $(h, C)$ be fuzzy soft right, lateral and left $N$-subgroups over $N$ respectively. Then $(f, A) \supseteq (g, B) \supseteq (h, C) = (k, D)$ is a fuzzy soft quasi-ideal over $N$.

Proof: Since $(f, A), (g, B)$ and $(h, C)$ are fuzzy soft right, lateral and left $N$-subgroups over $N$ they are fuzzy soft quasi-ideals and since $N$ is regular by the above theorem $(f, A) \supseteq (g, B) \supseteq (h, C) = (f, A) \supseteq (g, B) \supseteq (h, C)$. Hence by Corollary 4.8, $(f, A) \supseteq (g, B) \supseteq (h, C)$ is a fuzzy soft quasi-ideal over $N$.

5.14 Corollary

Let $N$ be a regular zero-symmetric RTNR and let $(f, A), (g, B)$ and $(h, C)$ be fuzzy soft right, lateral and left $N$-subgroups over $N$ respectively. Then $(f, A) \supseteq (g, B) \supseteq (h, C) = (k, D)$ is a fuzzy soft bi-ideal over $N$.

Proof: By the above corollary $(f, A) \supseteq (g, B) \supseteq (h, C) = (k, D)$ is a fuzzy soft quasi-ideal over $N$ and hence is a fuzzy soft bi-ideal by Lemma 5.4.

5.15 Lemma

Let $N$ be a regular zero-symmetric RTNR. Then

(i) $(f, A) = (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A)$ for every fuzzy soft bi-ideal $(f, A)$ over $N$.

(ii) $(f, A) = (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A)$ for every fuzzy soft quasi-ideal $(f, A)$ over $N$.

Proof: Let $N$ be a regular RTNR and $(f, A)$ be a fuzzy soft bi-ideal over $N$. Let $(h, A) = (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq ... (h, A)$. Then $(h, A) \subseteq (f, A)$. Let $x \in N$. Since $N$ is regular, there exists an element $u \in N$ such that $x = [ux]u$ i.e. $x = [x, u]u$. Now $h_1(x) = \min\{f_2(x), g_3(u), h_4(u)\} \supseteq f_1(x)$. Hence $(f, A) \subseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A)$. Thus $(f, A) = (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A)$. (ii) If $(f, A)$ is a fuzzy soft quasi-ideal then $(f, A)$ is a fuzzy soft bi-ideal and hence by (i) the proof follows.

5.16 Corollary

Let $N$ be a regular zero-symmetric RTNR. Then

(i) $(f, A) = (f, A) \supseteq (1, E) \supseteq (f, A)$ for every fuzzy soft bi-ideal $(f, A)$ over $N$.

(ii) $(f, A) = (f, A) \supseteq (1, E) \supseteq (f, A)$ for every fuzzy soft quasi-ideal $(f, A)$ over $N$.

Proof: (i) Let $N$ be a regular RTNR and $(f, A)$ be a fuzzy soft bi-ideal over $N$. Then by (i) of the above theorem $(f, A) = (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (f, A) \supseteq (1, E) \supseteq (f, A) \supseteq (f, A) \supseteq (f, A) = (f, A)$.

(ii) If $(f, A)$ is a fuzzy soft quasi-ideal then $(f, A)$ is a fuzzy soft bi-ideal and hence by (i) the proof follows.

5.17 Lemma

If $N$ is a regular zero-symmetric RTNR then

(i) $(f, A) \supseteq (g, B) = (f, A) \supseteq (g, B) \supseteq (f, A)$, where $(f, A)$ and $(g, B)$ are respectively fuzzy soft quasi-ideal and fuzzy soft lateral $N$-subgroup.
(ii) \((f, A) \uparrow (g, B) = (f, A) \uparrow (g, B) \uparrow (f, A)\), where \((f, A)\) and \((g, B)\) are respectively fuzzy soft bi-ideal and fuzzy soft lateral N-subgroup.

**Proof:** Let \(N\) be a regular RTNR and \((f, A), (g, B)\) be respectively fuzzy soft quasi-ideal and fuzzy soft lateral N-subgroup. By Corollary 5.16 (ii) \((f, A) = (f, A) \uparrow (1, E) \uparrow (f, A)\). Consider \((f, A) \uparrow (g, B) \uparrow (f, A) \uparrow (1, E) \uparrow (f, A) = (f, A)\). Similarly \((f, A) \uparrow (g, B) \uparrow (f, A) \subseteq (1, E) \subseteq (g, B) \subseteq (1, E) = (g, B)\), by Proposition 3.4(iv). Thus \((f, A) \uparrow (g, B) \uparrow (f, A) \subseteq (f, A) \uparrow (g, B).\) Now let \(x \in N\). Then as \(N\) is regular there exists \(u \in N\) such that \(x = [uxux]\). Consider \((f, A) \uparrow (g, B) \uparrow (f, A) = (f, A) = (h, C)\) where \(C = A \cap B\).

Now \(h(x) = \frac{\sum_{xyz} f_x(x) \land g_y([uxux]) \land f_z(x)}{f_z(x) \land g_y(x)} \geq f_x(x) \land g_y(x)\), as \((g, B)\) is a fuzzy soft lateral N-subgroup over \(N\). Hence \((f, A) \uparrow (g, B) \uparrow (f, A) \subseteq (f, A) \uparrow (g, B) \uparrow (f, A)\). Hence \((f, A) \uparrow (g, B) \uparrow (f, A) = (f, A) \uparrow (g, B) \uparrow (f, A)\).

(ii) If \((f, A)\) is a fuzzy soft quasi-ideal then \((f, A)\) is a fuzzy soft bi-ideal and hence by (i) the proof follows.

**5.18 Theorem**

Let \(N\) be an RTNR. Then \(N\) is left regular iff every fuzzy soft left N-subgroup is fuzzy soft completely semi-prime.

**Proof:** Let \(N\) be a left regular RTNR and \((f, A)\) be a fuzzy soft left N-subgroup over \(N\). Consider \(f_x(x) = f_x([uxx]) = f_x([uuxx]) \subseteq f_x(x)\), as \((f, A)\) is a fuzzy soft left N-subgroup over \(N\). Again since \((f, A)\) is a fuzzy soft left N-subgroup over \(N\), \(f_x(x) \subseteq f_x([uxx]) \subseteq f_x(x)\). Thus \(f_x(x) = f_x(x)\). Hence by Lemma 2.20, \((f, A)\) is fuzzy soft completely semi-prime.

Conversely, let every fuzzy soft left N-subgroup \((f, A)\) be fuzzy soft completely prime. Then by Lemma 2.20 \(f_x(x) = f_x(x)\) for any \(x \in N\). Since \([Nxx]\) is a left N-subgroup of \(N\) by Theorem 3.4(ii) \((\Psi_{N[xx]}(x \uparrow) = (\Psi_{N[xx]}(x \uparrow) \uparrow (x \uparrow) = 1 \Rightarrow (\Psi_{N[xx]}(x \uparrow) \Rightarrow x \in [Nxx] = x = [uxx]\) for some \(u \in N\) and \(v \in N\). Hence \(N\) is left regular.

**5.19 Lemma**

If \((f, A), (g, B), (h, C)\) are respectively fuzzy soft left, lateral and right N-subgroups over an intra-regular zero-symmetric RTNR \(N\) respectively then \((f, A) \uparrow (g, B) \uparrow (h, C) = (f, A) \uparrow (g, B) \uparrow (h, C)\).

**Proof:** Suppose that \(N\) is an intra-regular RTNR. By Theorem 4.8 (f, A) \uparrow (g, B) \uparrow (h, C) \subseteq (f, A) \uparrow (g, B) \uparrow (h, C)\). Let \(D = A \cap B \cap C\). Since \(N\) is intra-regular, \(x = [uxx]\) for some \(u, v \in N\). This implies that \(x = [uxx] = [uxx][uxx][uxx][uxx][uxx][uxx] = \ldots\). Consider \(f_x(g_y(h_z)(x) = \sum_{xyz} f_x(x) \land g_y(x) \land h_z(x) = \min\{f_x(x), g_y(x), h_z(x)\} = k(x)\). Thus \((f, A) \uparrow (g, B) \uparrow (h, C) \subseteq (f, A) \uparrow (g, B) \uparrow (h, C)\). Hence \((f, A) \uparrow (g, B) \uparrow (h, C) = (f, A) \uparrow (g, B) \uparrow (h, C)\).

**Theorem 5.20**

Let \(N\) be an intra-regular RTNR. Then

(i) A fuzzy soft set \((f, A)\) over \(N\) is a fuzzy soft N-subgroup over \(N\) if and only if \((f, A)\) is a fuzzy soft lateral N-subgroup over \(N\).

(ii) A fuzzy soft set \((f, A)\) over \(N\) is a fuzzy soft ideal over \(N\) if and only if \((f, A)\) is a fuzzy soft lateral ideal of \(N\).

**Proof:** It is obvious that if \((f, A)\) is a fuzzy soft N-subgroup over \(N\) then \((f, A)\) is a fuzzy soft lateral N-subgroup over \(N\). Conversely, let \((f, A)\) be a fuzzy soft lateral N-subgroup over \(N\). Let \(x, y, z \in N\), then there exists elements \(u, v \in N\) such that \(z = [uxx]\) and also \(x = [uxx]\) for some \(u, v \in N\). Now \(f_y([xyz]) = f_y([xyz][uxx]) = f_y([xy][z]) \supseteq f_y(z) \supseteq f_y(z)\). Also \(f_y([xyz]) = f_y([uxx][xyy]) = f_y([uxx][xyy]) \supseteq f_y(x) \supseteq f_y(x)\). Thus \((f, A)\) is a fuzzy soft N-subgroup over \(N\).

(ii) By Proposition 3.2 a fuzzy soft ideal over \(N\) is a fuzzy soft N-subgroup over \(N\) and hence by (i) the proof follows.

**6. CONCLUSIONS**

In this paper fuzzy soft N-subgroups, and quasi-ideals over an RTNR were introduced and their basic algebraic properties were studied. Fuzzy soft bi-ideals over a zero-symmetric RTNR were defined and their fundamental structure were explored. Left RTNRs were characterized in terms of fuzzy soft completely semi-prime left N-subgroups. A similar approach can be explored for non-zero-symmetric right ternary near-rings.

**REFERENCES**


