On the Vertex Covering Sets and Vertex Covering Polynomials of Square of Cycles

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ABSTRACT

In this paper, we derive the vertex covering polynomial for the square of cycle, $C_n^2$. The vertex covering polynomial of $G$, denoted by $C(G, x)$, is defined as $C(G, x) = \sum_{i=0}^{\Delta(G)} c(G, i)x^i$, where $c(G, i) = |C(G, i)|$. We obtain some properties of $C(C_n^2, x)$ and its coefficients. Also, we derive the reduction formula to calculate the vertex covering polynomials of square of cycle.

Keyword - square of cycle, vertex covering set, vertex covering number, vertex covering polynomial.

1. INTRODUCTION

Let $G = (V,E)$ be a graph. For any vertex $u \in V$, we define the open neighbourhood of $u$ as $N(u) = \{v \in V | uv \in E\}$ and the closed neighbourhood of $u$ as $N[u] = N(u) \cup \{u\}$. For a subset $S \subseteq V$, the open neighborhood of $S$ is $N(S)$ which is the union of $N(u)$ for all $u \in S$ and the closed neighborhood of $S$ is $N[S] \cup S$. The maximum degree of the graph $G$ is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A set $S \subseteq V$ is a vertex covering of $G$ if every edge $uv \in E$ is adjacent to at least one vertex in $S$. The vertex covering number, $\beta(G)$, is the minimum cardinality of the minimum vertex covering sets in $G$. A vertex covering set with cardinality $\beta(G)$ is called a $\beta$ set.

Definition 1.1

The second power of a graph $G$ is defined as the graph with the same set of vertices as $G$ and an edge between two vertices if and only if there is a path of length at most two between them. The second power of a graph is also called its square. Let $C_n^2$ be the square of the cycle $C_n$ with $n$ vertices and $E(C_n^2) = \{(1,2), (2,3), \ldots, (n-1,n), (1,3), (2,4), \ldots, (n-1,1), (n,2)\}$.

2. VERTEX COVERING SETS OF SQUARE OF THE CYCLE:

Definition 2.1

Let $C_n^2$ be the square of the cycle $C_n$ with $n$ vertices and $E(C_n^2)$. Let $C_n^2$ be the family of vertex covering sets of the square of the cycle $C_n$, with cardinality $\beta(C_n^2) = \beta(C_n) = \lfloor \frac{2n}{3} \rfloor$, $n \geq 4$. We use $[x]$ for the largest integer less than or equal to $x$ and $\lceil x \rceil$ for the smallest integer greater than or equal to $x$.

Lemma 2.2

Let $C_n^2$ be the square of the cycle $C_n$ with $n$ vertices. Then the vertex covering number of $C_n^2$ is $\beta(C_n^2) = \lfloor \frac{2n}{3} \rfloor$, $n \geq 4$.

Lemma 2.3

Let $C_n^2$ be the square of the cycle with $n$ vertices then $c(C_n^2, i) = 0$ if $i < \lfloor \frac{2n}{3} \rfloor$ or $i > n$ and $c(C_n^2, i) > 0$, if $\lfloor \frac{2n}{3} \rfloor \leq i \leq n$. Proof:

The minimum vertex covering number of the cycle $C_n^2$ is $\beta(C_n^2) = \lfloor \frac{2n}{3} \rfloor$.

Therefore, $c(C_n^2, i) = 0$, when $i < \lfloor \frac{2n}{3} \rfloor$ or $i > n$. Also, $c(C_n^2, i) > 0$ for $\lfloor \frac{2n}{3} \rfloor \leq i \leq n$.

Hence, we have $c(C_n^2, i) = 0$ if $i < \lfloor \frac{2n}{3} \rfloor$ or $i > n$ and $c(C_n^2, i) > 0$ if $\lfloor \frac{2n}{3} \rfloor \leq i \leq n$.
Lemma 2.4
Let $C_n^2$, $n \geq 4$ be the square of the cycle with $n$ vertices then

(i) $C(C_n^2, i) = \phi$ if $i < \left\lfloor \frac{2n}{3} \right\rfloor$ or $i > n$

(ii) $C(C_n^2, x)$ is a strictly increasing function on $[0, \infty]$.

(iii) $C(C_n^2, x)$ has no constant term and first degree terms.

Proof:
Proof is obvious.

Lemma 2.5
Let $C_n^2$ be the square of the cycle with $n$ vertices. Then

(i) If $C(C_n^2, i - 1) = C(C_n^2, i - 1) = \phi$ then $C(C_n^2, i - 1) = \phi$

(ii) If $C(C_n^2, i - 1) \neq \phi$ and $C(C_n^2, i - 2) \neq \phi$ then $C(C_n^2, i) \neq \phi$

(iii) If $C(C_n^2, i - 1) = C(C_n^2, i - 1) = C(C_n^2, i - 1) = \phi$ then $C(C_n^2, i) = \phi$

Proof:
(i) Let $C(C_n^2, i - 1) = C(C_n^2, i - 1) = \phi$, then

\[
\frac{2(n-1)}{3} > i - 1 \quad \text{or} \quad i-1 > n-1 \quad \text{and} \\
\frac{2(n-3)}{3} > i - 1 \quad \text{or} \quad i-1 > n-3
\]

From (1) and (2) we have $i-1 < \frac{2(n-3)}{3}$ or $i-1 > n-1$. Therefore, we have $\frac{2(n-2)}{3} > i - 1$ or $i-1 > n-2$. Therefore, $C(C_n^2, i - 1) = \phi$.

(ii) Given $C(C_n^2, i - 1) \neq \phi$ and $C(C_n^2, i-2) \neq \phi$.

Therefore, $\frac{2(n-1)}{3} \leq i - 1 \leq n - 1$ and $\frac{2(n-3)}{3} \leq i - 1 \leq n - 3$

Therefore, $\frac{2n-2}{3} \leq i - 1 \leq n - 3$; $\frac{2n-2}{3} + 1 \leq i \leq n - 2 \leq n$.

Therefore, $\frac{2n}{3} \leq i \leq n$. Hence $C(C_n^2, i) \neq \phi$.

(iii) By hypothesis, $i - 1 < \frac{2(n-1)}{3}$ or $i-1 > n-1$ and $i - 1 < \frac{2(n-2)}{3}$ or $i-1 > n-2$ and $i - 1 < \frac{2(n-3)}{3}$ or $i-1 > n-3$.

Therefore, $i - 1 < \frac{2n-6}{3}$ or $i-1 > n-1$; Therefore, $i < \frac{2n}{3}$ or $i > n$

Therefore, $C(C_n^2, i) = \phi$.

Theorem 2.6
Let $C_n^2$, $n \geq 4$ be the square of the cycle with $n$ vertices. Suppose that

$C(C_n^2, i) \neq \phi$ then we have

(i) $C(C_n^2, i - 1) \neq \phi$ and $C(C_n^2, i - 2) = \phi$ if and only if $n=3k+1, i = 2k+1$

(ii) $C(C_n^2, i - 1) = C(C_n^2, i - 2) = \phi$ and $C(C_n^2, i - 1) \neq \phi$ if and only if $i = n$.

(iii) $C(C_n^2, i - 1) \neq \phi$, $C(C_n^2, i - 1) \neq \phi$ and $C(C_n^2, i - 1) = \phi$

if and only if $i = n - 1$.

(iv) $C(C_n^2, i - 1) = \phi$, $C(C_n^2, i - 1) \neq \phi$ and $C(C_n^2, i - 1) \neq \phi$

if and only if $i = 3k$ and $i = 2k$ for some $k$.

(v) $C(C_n^2, i - 1) \neq \phi$, $C(C_n^2, i - 1) \neq \phi$; $C(C_n^2, i - 1) \neq \phi$ if and only if $\frac{2(n-1)}{3} \leq i - 1 \leq n - 3$.

Proof:
(i) Assume \( \mathcal{C}(C_{n+1}^{2}, i-1) \neq \phi \), and \( \mathcal{C}(C_{n+2}^{2}, i-2) = \phi \). Then \( i-2 > n-2 \) (or) \( i-2 < \left\lfloor \frac{2(n-2)}{3} \right\rfloor \). Suppose \( i-2 > n-2 \) then \( i > n \). Therefore, \( \mathcal{C}(C_{n+1}^{2}, i) = \phi \), which is a contradiction. If
\[
i-2 < \left\lfloor \frac{2(n-2)}{3} \right\rfloor \text{ then } i < \left( \frac{2n-4}{3} \right) + 2
\]
Also, \( \mathcal{C}(C_{n+1}^{2}, i-1) \neq \phi \). Therefore, \( \left\lfloor \frac{2(n-1)}{3} \right\rfloor \leq i - 1 \leq n - 1 \)

Therefore, \( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \leq i \leq n \).

From (3) & (4) \( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{2n-4}{3} \right\rfloor + 2 \).

If \( n \neq 3k+1 \), then from (5), we obtain an inequality of the form
\( s \leq i < s \) which is not possible. When \( n = 3k+1 \) then (5) holds and in this case \( i=2k+1 \). Conversely, assume \( n=3k+1 \) and \( i=2k+1 \). Therefore \( n-2 = 3k-1 \) and \( 2(n-2)= 6k -2.\) Therefore, \( \frac{2(n-2)}{3} = 2k - 1 = i - 2 \).

Therefore, \( i-2 < \left\lfloor \frac{2(n-2)}{3} \right\rfloor \). Therefore, \( \mathcal{C}(C_{n+2}^{2}, i-2) = \phi \).

Also, \( n-1 = 3k \) which implies \( \frac{n-1}{3} = k \)

And \( i-1 = 2k \). This implies \( \frac{i-1}{2} = k \)

From (6) & (7) \( \frac{n-1}{3} = \frac{i-1}{2} \). That is \( \frac{2(n-1)}{3} = i-1 \)

Therefore \( \left\lfloor \frac{2(n-1)}{3} \right\rfloor \leq i - 1 \leq n - 1 \). Which implies \( \mathcal{C}(C_{n-1}^{2}, i-1) \neq \phi \)

(ii) Assume \( \mathcal{C}(C_{n+1}^{2}, i-1) = \phi \), and \( \mathcal{C}(C_{n+3}^{2}, i-2) = \phi \). Then
we have \( i-1 > n-2 \) (or) \( i-1 < \left\lfloor \frac{2(n-2)}{3} \right\rfloor \) and \( i-2 < \left\lfloor \frac{2(n-3)}{3} \right\rfloor \) (or) \( i-2 > n-3 \).

Therefore, \( i-1 < \left\lfloor \frac{2n-6}{3} \right\rfloor \) or \( i-1 > n-2 \). Suppose \( i-1 < \left\lfloor \frac{2n-6}{3} \right\rfloor \), then \( i-1 < \left\lfloor \frac{2n-2}{3} \right\rfloor \).

Therefore, \( \mathcal{C}(C_{n+3}^{2}, i-1) = \phi \) which is a contradiction. Therefore, \( i-1 > n-2 \).

That is \( i > n-1 \) which implies that \( i \geq n \). Also, since \( \mathcal{C}(C_{n}^{2}, i) \neq \phi \) we have \( i \leq n \) combining these we get \( i=n \).

Conversely, if \( i=n \), then \( \mathcal{C}(C_{n+2}^{2}, i-1) = \mathcal{C}(C_{n+2}^{2}, n-1) = \phi \).

Since \( n-1 > n-2 \), \( \mathcal{C}(C_{n+3}^{2}, i-2) = \mathcal{C}(C_{n+3}^{2}, n-2) = \phi \) and
\( \mathcal{C}(C_{n+1}^{2}, i-1) = \mathcal{C}(C_{n+1}^{2}, n-1) \neq \phi \).

(iii) Since \( \mathcal{C}(C_{n+2}^{2}, i-1) = \phi \), we have \( i-1 > n-3 \) (or) \( i-1 < \left\lfloor \frac{2(n-3)}{3} \right\rfloor \)

Since \( \mathcal{C}(C_{n+1}^{2}, i-1) \neq \phi \), we have \( \left\lfloor \frac{2(n-1)}{3} \right\rfloor \leq i-1 \leq n-1 \)

Suppose \( i-1 < \left\lfloor \frac{2(n-3)}{3} \right\rfloor \) then (9) does not hold. Therefore, our assumption is wrong. Therefore, \( i-1 > n-3 \). Which implies that \( i-1 \geq n - 2 \)

Also, since \( \mathcal{C}(C_{n+2}^{2}, i-1) \neq \phi \), we have \( \left\lfloor \frac{2(n-2)}{3} \right\rfloor \leq i - 1 \leq n \)

From (10) & (11), we have \( i-1=n-2 \). Therefore, \( i = n-1 \). Conversely, assume \( i = n-1 \). Then \( \mathcal{C}(C_{n+1}^{2}, i-1) = \mathcal{C}(C_{n+1}^{2}, n-2) \neq \phi \) and
\( \mathcal{C}(C_{n-2}^{2}, i-1)=\mathcal{C}(C_{n-2}^{2}, n-2) \neq \phi \) and \( \mathcal{C}(C_{n+3}^{2}, i-1) = \mathcal{C}(C_{n+3}^{2}, n-2) \neq \phi \). Since \( n-2 > n-3 \).

(iv) Since \( \mathcal{C}(C_{n+2}^{2}, i-1) = \phi \), by lemma 2.2, \( i-1 > n-1 \) or \( i-1 < \left\lfloor \frac{2(n-1)}{3} \right\rfloor \).

Suppose \( i-1 > n-1 \), then \( i-1 > n-2 \). Therefore, \( \mathcal{C}(C_{n+2}^{2}, i-1) = \phi \) which is a contradiction.

Therefore, \( i-1 < \left\lfloor \frac{2(n-1)}{3} \right\rfloor \)

Since \( \mathcal{C}(C_{n+2}^{2}, i-1) \neq \phi \), we have \( \left\lfloor \frac{2(n-2)}{3} \right\rfloor \leq i-1 \leq n-2 \)
Also, since \( \mathcal{C}(C_{n-3}^2, i-1) \neq \phi \), we have \( \frac{2(n-3)}{3} \leq i - 1 \leq n - 3 \) \( (14) \)

Since \( \mathcal{C}(C_n^2, i) \neq \phi \), we have \( \left\lfloor \frac{2n}{3} \right\rfloor \leq i \leq n \)

Therefore, \( \frac{2n}{3} - 1 \leq i - 1 \leq n - 1 \) \( (15) \)

Combining (12) to (15), we have \( \frac{2n}{3} \leq i < \frac{2(n-2)}{3} + 1 \).

when \( n \neq 3k \) we get an inequality of the form \( \leq i - 1 \leq t \) which is not possible. When \( n=3k \), (15) holds and in this case \( i=2k \). Conversely, assume \( n=3k \) and

\( i = 2k \). We have \( n-1=3k-1 \) and \( i-1 = 2k-1 \). Then \( \left\lfloor \frac{2n-2}{3} \right\rfloor = \left\lfloor \frac{6k-2}{3} \right\rfloor = 2k > 2k-1-i-1 \).

Therefore, \( i-1 < \frac{2(n-1)}{3} \). Therefore, \( \mathcal{C}(C_{n-1}^2, i-1) = \phi \). Also \( n=2 = 3k-2 \) and

\( i-1 = 2k-1 \). That is \( 2(n-2) = 6k-4 \) and \( 3(i-1) = 6k-3 \). Now \( 2(n-2) = 6k-4 < 6k-3 \). Therefore, \( 2(n-2) < 3(i-1) \).

That is \( \frac{2(n-2)}{3} < i-1 = 2k-1 \leq 3k-2 \). Hence

\( \frac{2(n-2)}{3} \leq i - 1 \leq n - 2 \). Therefore, \( \mathcal{C}(C_{n-2}^2, i-1) \neq \phi \).

Also \( n-3 \) \( = 3k-3 \) and \( i-1 = 2k-1 \). That is \( n-3 = 3(k-1) \) and \( i-2 = 2k-2 \). Therefore \( \frac{n-3}{3} = k-1 \) and \( \frac{i-2}{2} = k-1 \).

Therefore, \( \frac{n-3}{3} = \frac{i-2}{2} \), which implies that \( \frac{2(n-3)}{3} = i-2 < i-1 \). Hence we have \( \frac{2(n-3)}{3} \leq i-1 \leq n-3 \).

Since \( 2k-1 \leq 3k-3 \), we have \( \frac{2(n-3)}{3} \leq i-1 \leq n-3 \). Therefore, \( \mathcal{C}(C_{n-3}^2, i-1) \neq \phi \).

\( \text{(v) Assume } \mathcal{C}(C_{n-1}^2, i-1) \neq \phi, \mathcal{C}(C_{n-2}^2, i-1) \neq \phi \) and

\( \mathcal{C}(C_{n-3}^2, i-1) \neq \phi \). Then we have \( \frac{2(n-1)}{3} \leq i-1 \leq n-1 \),

\( \frac{2(n-2)}{3} \leq i-1 \leq n-2 \) and \( \frac{2(n-3)}{3} \leq i-1 \leq n-3 \). Hence we have \( \frac{2(n-1)}{3} \leq i-1 \leq n-3 \). Converse part is obvious.

**Theorem 2.7**

For every \( n \geq 4 \) and \( i \geq \left\lfloor \frac{2n}{3} \right\rfloor \), we have

i. If \( \mathcal{C}(C_{n-1}^2, i-1) \neq \phi \) and \( \mathcal{C}(C_{n-3}^2, i-2) = \phi \),

then \( \mathcal{C}(C_n^2, i) = \mathcal{C}(C_n^2, n) = \{1, 2, 3, \ldots, n\} \).

ii. If \( \mathcal{C}(C_{n-1}^2, i-1) \neq \phi \), \( \mathcal{C}(C_{n-2}^2, i-1) \neq \phi \) and \( \mathcal{C}(C_{n-3}^2, i-1) = \phi \), then

\( \mathcal{C}(C_n^2, i) = \mathcal{C}(C_n^2, n-1) = \{n\} \cap \{x \mid x \in [n]\} \).

iii.

\( \mathcal{C}(C_{n+1}^2, 2n) = \left\{ \begin{array}{ll}
2, 3, 5, 6, \ldots, 3n - 4, 3n - 3, 3n - 1, 3n, & \\
1, 3, 4, 6, \ldots, 3n - 5, 3n - 3, 3n - 2, 3n, & \\
1, 2, 4, 5, \ldots, 3n - 5, 3n - 4, 3n - 2, 3n - 1, &
\end{array} \right. \}

iv. If \( \mathcal{C}(C_{n-1}^2, i-1) \neq \phi \) and \( \mathcal{C}(C_{n-3}^2, i-2) \neq \phi \) then

\( \mathcal{C}(C_n^2, i) = \left\{ \begin{array}{ll}
\{X \cup \{n\} \mid X \in \mathcal{C}(C_{n-1}^2, i-1) \} \cup & \\
\{Y \cup \{n, n-1\} \mid Y \in \mathcal{C}(C_{n-3}^2, i-2) \text{ if } n-5, n-4, n-3 \in Y \} \cup & \\
\{Y \cup \{n-1, n-2\} \mid Y \in \mathcal{C}(C_{n-3}^2, i-2) \text{ if } n-6, n-5, n-4 \in Y \} \cup & \\
\{Y \cup \{n-2\} \mid Y \in \mathcal{C}(C_{n-3}^2, i-2) \text{ if } n-6, n-5, n-3 \in Y \} \cup & \\
\{Y \cup \{n-1, n-2\} \mid Y \in \mathcal{C}(C_{n-3}^2, i-2) \text{ if } n-6, n-4, n-3 \in Y \} &
\end{array} \right. \} \)
Proof:

(i) Since \( C(C_{n-1}^2, i - 1) \neq \emptyset \) and \( C(C_{n-3}^2, i - 2) = \emptyset \) by theorem 2.6 (ii), \( i = n \). Therefore, \( C(C_{n}^2, i) = C(C_{n}^2, n) = \{1, 2, 3, \ldots, n\} \).

(ii) If \( C(C_{n-1}^2, i - 1) \neq \emptyset \) and \( C(C_{n-2}^2, i - 1) \neq \emptyset \) and \( C(C_{n-3}^2, i - 1) = \emptyset \), then by theorem 2.6 (iii), \( i = n - 1 \). Therefore \( C(C_{n}^2, i) = C(C_{n}^2, n - 1) = \{n\} \).

(iii) For any \( n \geq 4 \), \( C(C_{2n}^2, 2n) \) has only three vertex covering sets as

\[
C(C_{2n}^2, 2n) = \left\{ \begin{array}{l}
\{2, 3, 5, 6, \ldots, 3n - 4, 3n - 3, 3n - 1, 3n\} \\
\{1, 3, 4, 6, \ldots, 3n - 5, 3n - 3, 3n - 2, 3n\} \\
\{1, 2, 4, 5, \ldots, 3n - 5, 3n - 4, 3n - 2, 3n - 1\}
\end{array} \right\}
\]

(iv) The construction of \( C(C_{n}^2, i) \) from \( C(C_{n-1}^2, i - 1) \) and \( C(C_{n-3}^2, i - 2) \) is as follows: Let X be the vertex covering set of \( C(C_{n-1}^2, i - 1) \) with cardinality \( i - 1 \). All the elements of \( C(C_{n-1}^2, i - 1) \) end with \( n - 1 \) or \( n - 2 \). Therefore, adjoin \( n \) with X. Hence if \( X \in C(C_{n-1}^2, i - 1) \) then \( X \cup \{n\} \in C(C_{n}^2, i) \). Next, let us consider \( C(C_{n-3}^2, i - 2) \). Here, all the elements of \( C(C_{n-3}^2, i - 2) \) end with \( n - 3 \) or \( n - 4 \) or both. Let Y be the vertex covering set of \( C(C_{n-3}^2, i - 2) \) with cardinality \( i - 2 \). If \( n - 5, n - 4, n - 3 \) belong to \( Y \) then \( Y \cup \{n, n - 1\} \in C(C_{n}^2, i) \). If \( n - 6, n - 5, n - 4 \) belong to \( Y \) then \( Y \cup \{n, n - 1, n - 2\} \in C(C_{n}^2, i) \). If \( n - 6, n - 4 \) and \( n - 3 \) belong to \( Y \) then \( Y \cup \{n - 1, n - 2\} \in C(C_{n}^2, i) \). Therefore, we cover all the elements of \( C(C_{n}^2, i) \) by means of the elements of \( C(C_{n-1}^2, i - 1) \) and \( C(C_{n-3}^2, i - 2) \). Therefore, \( C(C_{n}^2, i) \subseteq \text{R.H.S.} \)

Converse part is obvious. Therefore,

\[
C(C_{n}^2, i) = \begin{cases}
\{X \cup \{n\} / X \in C(C_{n-1}^2, i - 1)\} \cup \\
\{Y \cup \{n, n - 1\} / Y \in C(C_{n-3}^2, i - 2) \text{ if } n - 5, n - 4, n - 3 \in Y \} \cup \\
\{Y \cup \{n, n - 1, n - 2\} / Y \in C(C_{n-3}^2, i - 2) \text{ if } n - 6, n - 5, n - 4 \in Y \} \cup \{YU \{n - 1, n - 2\} Y \in C(C_{n-3}^2, i - 2) \text{ if } n - 6, n - 4, n - 3 \in Y \}
\end{cases}
\]

Theorem 2.8

If \( C(C_{n}^2, i) \) is the family of vertex covering set of \( C_{n}^2 \) with cardinality \( i \), where \( i \geq \frac{2n}{3} \) then \( c(C_{n}^2, i) = c(C_{n-1}^2, i - 1) + c(C_{n-3}^2, i - 2) \).

Proof:

From theorem 2.7, we consider all the three cases as given below where \( i > \frac{2n}{3} \)

(i) If \( C(C_{n}^2, i - 1) = C(C_{n-1}^2, i - 1) = \emptyset \), then \( C(C_{n}^2, i) = \emptyset \).

(ii) If \( C(C_{n}^2, i - 1) \neq \emptyset \) and \( C(C_{n-3}^2, i - 2) = \emptyset \), then \( C(C_{n}^2, i) = \{X \cup \{n\} / X \in C(C_{n-1}^2, i - 1)\} \).

(iii) If \( C(C_{n}^2, i - 1) \neq \emptyset \) and \( C(C_{n-3}^2, i - 2) \neq \emptyset \),

\[
C(C_{n}^2, i) = \begin{cases}
\{X \cup \{n\}\} \cup \\
\{Y \cup \{n, n - 1\} / n - 5, n - 4, n - 3 \in Y\} \cup \\
\{Y \cup \{n, n - 1, n - 2\} / n - 6, n - 5, n - 4 \in Y\} \cup \\
\{YU \{n, n - 2\} / n - 6, n - 5, n - 3 \in Y\} \cup \{YU \{n - 1, n - 2\} / n - 6, n - 4, n - 3 \in Y\}
\end{cases}
\]

From the above construction in each case we obtain that

\[
|C(C_{n}^2, i)| = |C(C_{n-1}^2, i - 1)| + |C(C_{n-3}^2, i - 2)|.
\]

That is \( c(C_{n}^2, i) = c(C_{n-1}^2, i - 1) + c(C_{n-3}^2, i - 2) \).

3. VERTEX COVERING POLYNOMIAL OF SQUARE OF THE CYCLE

Let \( C(C_{n}^2, x) \) be the vertex covering sets of \( C_{n}^2 \) with cardinality \( i \) and let \( c(C_{n}^2, i) = |C(C_{n}^2, i)| \). Then the vertex covering polynomial, \( C(C_{n}^2, x) \) is defined as \( C(C_{n}^2, x) = \sum_{i=1}^{\lfloor \frac{2n}{3} \rfloor} c(C_{n}^2, i) x^i \).
In this section we derive the expression for $C(C_n^2,x)$.

**Theorem 3.1**

For every $n \geq 6$, $C(C_n^2,x) = xC(C_{n-1}^2,x) + x^2C(C_{n-3}^2,x)$ with initial values $C(C_4^2,x) = 4x^3 + x^4$, $C(C_5^2,x) = 5x^4 + x^5$.

**Proof**

We have $c(C_n^2, i) = c(C_{n-1}^2, i - 1) + c(C_{n-3}^2, i - 2)$. Therefore,

$$
c(C_n^2, i) x^i = c(C_{n-1}^2, i - 1)x^i + c(C_{n-3}^2, i - 2)x^i
$$

$$
\sum c(C_n^2, i) x^i = \sum c(C_{n-1}^2, i - 1)x^i + \sum c(C_{n-3}^2, i - 2)x^i
$$

$$
\sum c(C_n^2, i)x^i = x \sum c(C_{n-1}^2, i - 1)x^{i+1} + x^2 \sum c(C_{n-3}^2, i - 2)x^{i+2}
$$

$C(C_n^2,x) = xC(C_{n-1}^2,x) + x^2C(C_{n-3}^2,x)$ with initial values $C(C_4^2,x) = 4x^3 + x^4$.

$C(C_5^2,x) = 5x^4 + x^5$.

We obtain $c(C_n^2, i)$ for $4 \leq n \leq 16$ as shown in the table 1.

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In the following theorem we obtain some properties of $c(C_n^2, i)$. 
Theorem: 3.2

The following properties hold for the coefficients of \( C(C_n^2, x) \).

(i) For \( 4 \leq n \), \( c(C_n^2, n) = 1 \)

(ii) For \( 4 \leq n \), \( c(C_n^2, n - 1) = n \)

(iii) For \( 6 \leq n \), \( c(C_n^2, n - 2) = \frac{1}{2} [n^2 - 5n] \)

(iv) For \( 2 \leq k, c(C_{3k}^2, 2k) = 3 \)

(v) For \( 2 \leq k, c(C_{3k-2}^2, 2k - 1) = 3k-2 \)

(vi) For \( 1 \leq k, c(C_{3k+1}^2, 2k + 1) = 3k+1 \) for all \( k \in \mathbb{N} \).

(vii) For \( 1 \leq k, c(C_{3k+2}^2, 2(k + 1)) = \frac{3k^2+5k+2}{2} \)

Proof:

(i) Since, for any graph with \( n \) vertices the vertex covering number \( c(G,n)=1 \) then \( c(C_n^2, n) = 1 \).

(ii) Since \( c(C_n^2, n - 1) = \{[n] - \{x\} / x \in [n]\} \), we have \( c(C_n^2, n - 1) = n \).

(iii) To prove \( c(C_n^2, n - 2)=\frac{1}{2} [n^2 - 5n] \), \( n\geq 6 \) we apply induction on \( n \).

When \( n=6 \).

L.H.S= \( c(C_6^2, 6 - 2) = c(C_4^2, 4) = 3 \) (from the table)

R.H.S = \( \frac{1}{2} [n^2 - 5n] = \frac{1}{2} [36 - 30] = \frac{1}{2} [6] = 3 \).

Therefore, the result true for \( n = 6 \). Now suppose that the result is true for all numbers less than ‘n’ and we prove it for \( n \).

\[ c(C_n^2, n - 2) = c(C_{n-1}^2, n - 3) + c(C_{n-3}^2, n - 4) = \frac{1}{2} [(n - 1)^2 - 5(n - 1)] + n-3 \]

\[ = \frac{1}{2} [n^2 + 1 - 2n - 5n + 5] + n - 3 = \frac{1}{2} [n^2 + 1 - 2n - 5n + 5] + n - 3 \]

\[ = \frac{1}{2} [n^2 - 7n + 6 + 2n - 6] = \frac{1}{2} [n^2 - 5n]. \]

Hence the result is true for all \( n \). Hence by the induction hypothesis we have, \( c(C_n^2, n - 2)=\frac{1}{2} [n^2 - 5n] \), \( \forall n \geq 6 \).

(iv) To prove \( c(C_{3k}^2, 2k) = 3 \) From theorem (2.7) we have

\[ \begin{align*}
2,3,5,6 \ldots \ldots \ldots 3k-4,3k-3,3k-1,3k \\
1,3,4,6, \ldots \ldots \ldots 3k-5,3k-3,3k-2,3k
\end{align*} \]

\[ C(C_{3k}^2, 2k) = \begin{cases}
\{2,3,5,6 \ldots \ldots \ldots 3k-4,3k-3,3k-1,3k\} \\
\{1,3,4,6, \ldots \ldots \ldots 3k-5,3k-3,3k-2,3k\}
\end{cases} \]

\[ \therefore c(C_{3k}^2, 2k) = \]

that is \( c(C_{3k}^2, 2k) = 3 \), \( \forall k \geq 2 \).

(v) To prove \( c(C_{3k-2}^2, 2k - 1) = 3k-2 \), \( k \geq 2 \).

We apply induction on \( k \).

Let \( k = 2 \).

L.H.S: \( c(C_{3k-2}^2, 2k - 1) = c(C_4^2, 3) = 4 \) (from the table).

R.H.S: \( 3k-2 = 3x2-2=6 - 2 = 4 \). Therefore, \( c(C_{3k-2}^2, 2k - 1) = 3k-2 \) is true for \( k = 2 \).

Assume the result is true for all natural numbers less than \( k \) and we will prove it for \( k \).

\[ c(C_{3k-2}^2, 2k - 1) = c(C_{3k-3}, 2k - 2) + c(C_{3k-2}^2, 2k - 3) \]

\[ = c(C_{3(k-1)}^2, 2(k-1) + c(C_{3(k-1)}^2, 2(k-1) - 1) = 3 + 3k - 5 = 3k - 2. \]
\[ \therefore c \left( C_{3k-2}^2, 2k - 1 \right) = 3k - 2 \forall k \geq 2. \]

(vi) To prove \( c \left( C_{3k+1}^2, 2k + 1 \right) = 3k + 1 \), \( k \geq 1 \) we apply induction on \( k \).
Let \( k = 1 \). L.H.S: \( c \left( C_{3k+1}^2, 2k + 1 \right) = c \left( C_6^2, 3 \right) = 4 \).
R.H.S: \( 3k + 1 = 3 + 1 = 4 \). Therefore, the result is true for \( k = 1 \).
Assume the result is true for all natural numbers less than \( k \), and we prove it for \( k \).
\[ c \left( C_{3k+1}^2, 2k + 1 \right) = c \left( C_{3k}^2, 2k \right) + c \left( C_{3k-2}^2, 2k - 1 \right) = 3 + 3k - 2 = 3k - 1. \]
Therefore, \( c \left( C_{3k+1}^2, 2k + 1 \right) = 3k + 1 \) for all \( k \geq 1 \).

(vii) To prove \( c \left( C_{3k+2}^2, 2(k + 1) \right) = \frac{1}{2} \left[ 3k^2 + 5k + 2 \right] \forall k \geq 1 \).
We apply induction on \( k \). Let \( k = 1 \).
Then L.H.S: \( c \left( C_{3k+2}^2, 2(k + 1) \right) = c \left( C_6^2, 4 \right) = 5 \) (from table 1)
R.H.S: \( \frac{1}{2} \left[ 3k^2 + 5k + 2 \right] = \frac{1}{2} [3 + 5 + 2] = \frac{1}{2} [10] = 5 \).
Therefore, the result is true for \( k = 1 \).
Assume the result is true for all natural numbers less than \( k \), and we prove it for \( k \).
\[ c \left( C_{3k+2}^2, 2(k + 1) \right) = c \left( C_{3(k+2)-1}^2, 2(k + 1) - 1 \right) + c \left( C_{3(k+2)-3}^2, 2(k + 1) - 2 \right) \]
\[ = c \left( C_{3k+1}^2, 2k + 1 \right) + c \left( C_{3k-1}^2, 2k \right) \]
\[ = c \left( C_{3k+1}^2, 2k + 1 \right) + c \left( C_{3(k-1)+2}^2, 2((k - 1) + 1) \right) \]
\[ = 3k + 1 + \frac{1}{2} \left[ 3(k-1)^2 + 5(k-1) + 2 \right] = 3k + 1 + \frac{1}{2} \left[ 2k^2 - 1 - 2k \right] + 5k - 5 + 2 \]
\[ = 3k + 1 + \frac{1}{2} \left[ 3k^2 + 3 - 6k + 5k - 3 \right] = 3k + 1 + \frac{1}{2} \left[ 3k^2 - k \right] \]
\[ = \frac{1}{2} \left[ 3k^2 - k + 6k + 2 \right] = \frac{1}{2} \left[ 3k^2 + 5k + 2 \right]. \] Hence the result is true for all \( k \geq 1 \).
Therefore, \( c \left( C_{3k+2}^2, 2(k + 1) \right) = \frac{1}{2} \left[ 3k^2 + 5k + 2 \right] \).

4. CONCLUSION
In this paper the vertex cover polynomial of square of cycle has been derived by identifying its vertex covering sets. It also helps us to characterize the vertex covering sets and to find the number of vertex covering sets of cardinality \( i \). We can generalize this study to any power of the cycle and some interesting properties can be obtained via the roots of vertex cover polynomial of \( C_n^k \).

REFERENCES