Inequalities on Hardy and Higher Power Weighted Bergman Spaces of Composition Operators

Elhadi Elnour Elniel
Al Baha University, Al Mandag, Saudi Arabia, Department of Mathematics, College of Arts and Science, Email:elhadielniel_2003@hotmail.com

Shawgy Hussein Abd Allah
Sudan University of Sciences and Technology, Khartoum, Sudan, Department of Mathematics, College of Science, Email:shha2020@gmail.com

ABSTRACT

Bounded Composition operators usually induced by an analytic self-map of the open unit disk on the Hardy space $H^2$ and on the higher power weighted Bergman spaces $L^2_{(\alpha+1)^2-1}$. An inequality for the relationship between the norms of the corresponding operators on these spaces is considered.

Keywords- Norm inequalities, composition operators, Hardy spaces, power weighted Bergman spaces, Schur Product Theorem.

1. INTRODUCTION

Let $D$ denote the open unit disk in the complex plane and let $\varphi$ be an analytic self-map of $D$. If $H$ is a Hilbert space of analytic functions on $D$, the composition operator $C_\varphi$ on $H$ is defined by the rule $C_\varphi(f) = f \circ \varphi$. While there are some Hilbert spaces (the Dirichlet space) on which there are unbounded composition operators, every analytic $\varphi : D \to D$ induces a bounded operator on all of the spaces we will be considering in this paper. We Show the relationship between the operator norms of $C_\varphi$ acting on different spaces with weights.

The Hilbert spaces of primary interest to us will be the Hardy space $H^2$ and the power weighted Bergman spaces $L^2_{(\alpha+1)^2-1}$. The Hardy space consists of all analytic functions $f$ on $D$ such that

$$\|f\|_{H^2}^2 = \frac{1}{2\pi} \sup_{0<\alpha<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty,$$

with inner product

$$<f,g>_{H^2} = \frac{1}{2\pi} \lim_{r \to 1} \int_0^{2\pi} f(r e^{i\theta}) g^{\alpha}(r e^{i\theta}) \, d\theta < \infty.$$

The Hardy space can be described as a reproducing Kernel Hilbert space, since for every point $\lambda$ in $D$ there is a unique function $K_\lambda$ in $H^2$ such that $\langle f,K_\lambda \rangle_{H^2} = f(\lambda)$ for all $f$ in $H^2$, and $K_\lambda(z) = \frac{1}{1 - \lambda z}$ (see [1]).

For $\alpha > -1$ we write the proposed higher power weighted Bergman space $L^2_{(\alpha+1)^2-1}$ which consists of all analytic functions $f$ on $D$ such that

$$\|f\|^2_{(\alpha+1)^2-1} = \int_D |f(z)|^2 (\alpha+1)^2 (1-|z|^{\alpha+1})^{(\alpha+1)^2-1} \, dA(z) < \infty,$$

where $dA$ signifies normalized area measure on $D$.

We write $\langle \cdot, \cdot \rangle_{(\alpha+1)^2-1}$, for any $\alpha$, to denote the inner product on $L^2_{(\alpha+1)^2-1}$ with kernel function $K_\lambda^{(\alpha+1)^2-1}(z) = \frac{1}{1 - \lambda z^{(\alpha+1)^2-1}}$. There is an obvious likeness between the reproducing kernels for $H^2$ and
the analogous functions for $L^2_{(\alpha+1)^2-1}$. For the sake of efficiency, we write $L^2_{\alpha}$ to denote the Hardy space $H^2_{\alpha}$, with $K_{\alpha}^{-1} = K_{\alpha}$ and $\ldots, L^2_{\alpha} = \ldots, H^2_{\alpha}$. We will state many of the results in these terms, with the understanding that $\alpha = 0$ and $\alpha = -2$ "power weighted Bergman spaces" always signifies the Hardy space.

For any analytic $\varphi : D \to D$, we will write $\|C_\varphi\|_{L^2}$ to denote the norm of $C_\varphi$ acting on a Hilbert space $H$. While it is generally not easy to calculate $\|C_\varphi\|_{L^2_{(\alpha+1)^2-1}}$ explicitly, (see [2], [3], [4], [5], [6], [7], and [8]). In fact, it is not difficult to estimate the norm of $C_\varphi$. In particular, it is well known that

$$\left(\frac{1}{1 - \|\varphi(0)\|^2}\right)^{(\alpha+1)^2+1} \leq \left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{(\alpha+1)^2+1}$$

For any $\alpha \geq -1$ (see [1], [9]). In spite of (1), one might wonder whether there is some relationship between the quantities $\|C_\varphi\|_{L^2}$ for different values of $\alpha$.

For example, considering $\alpha = 0$, $\alpha = -1$ and $\alpha = -2$, one might ask whether it is always the case that $\|C_\varphi\|_{L^2} = \|C_\varphi\|_{H^2}$. While this equality does hold for some maps, it is not true in general (see [10]). Christopher Hammond and Linda Patton[11] proved that $\|C_\varphi\|_{L^2} \leq \|C_\varphi\|_{H^2}$ for all $\varphi$ answering a question posed by Carswell and Hammond [10], and they derived a collection of inequalities relating to the norms of $C_\varphi$ on different spaces.

In this paper we apply Norm inequalities for Composition operators [11] to give a verification of higher power weighted Bergman spaces. Now we should mention a helpful fact relating to composition operators and reproducing kernel functions. Let $C^*_{\varphi}$ denote the adjoint of $C_\varphi$ on a particular space $L^2_{(\alpha+1)^2-1}$, and that $C^*_{\varphi}(k_{\lambda}^{(\alpha+1)^2-1}) = k_{\varphi(\lambda)}^{(\alpha+1)^2-1}$ for any $\lambda$ in $D$ (see [1]). This observation will provide exactly the verification of the information we need to compare the action of $C_\varphi$ on different spaces.

2. SEMIDEFINITE MATRICES

Let $\Lambda = \{\lambda_m\}_{m=1}^{\infty}$, a sequence of distinct points in $D$, be a set of uniqueness for the collection of analytic functions on $D$. In other words, the zero function is the only analytic function with $f(\lambda_m) = 0$ for all $m$. The span of the kernel functions $\{k_{\lambda}^{(\alpha+1)^2-1}\}_{m=1}^{\infty}$ is dense in every space $L^2_{(\alpha+1)^2-1}$, since any function orthogonal to each $k_{\lambda}^{(\alpha+1)^2-1}$ must be identically 0. Throughout this paper, we will assume that such a sequence $\Lambda$ has been fixed.

Consider an analytic map $\varphi : D \to D$. For a positive constant $\nu$, a natural number $n$, and a real number $\alpha \geq -1$, we define the $n \times n$ matrix by

$$M(\nu, n, \alpha(\alpha + 2)) = \left[\frac{\nu^2}{(1 - \lambda_j \lambda_i)(\alpha+1)^2+1} - \frac{1}{(1 - \nu(\lambda_j)(\varphi(\lambda_i))(\alpha+1)^2+1)}\right]_{j,i=1}^{n}$$

In particular
Recall that an \( n \times n \) matrix \( A \) is called positive semidefinite if \( \langle Ac, c \rangle \geq 0 \) for all \( c \) in \( \mathbb{C}^n \), location as \( A \geq 0 \) where \( \langle ., . \rangle \) denotes the standard Euclidean inner product. Any such matrix must necessarily be self-adjoint. For self-adjoint matrices \( A \) and \( B \), we write \( A \geq B \) to denote \( A - B \) being positive semidefinite. The following proposition relates \( \|C_\varphi\|_{\ell_2(\alpha^2)}^2 \) to the positive semidefiniteness of \( M(\nu, n, \alpha(\alpha + 2)) \).

### 2.1 Proposition

Let \( \varphi \) be an analytic self-map of \( D \) and \( \nu \) be a positive constant. Then, for any \( \alpha \geq -1 \), the matrix \( M(\nu, n, \alpha(\alpha + 2)) \) is positive semidefinite for all natural numbers \( n \) if and only if

\[
\|C_\varphi\|_{\ell_2(\alpha^2)}^2 \leq \nu.
\]

**Proof.** Assume first that

\[
\|C_\varphi(f)\|_{\ell_2(\alpha^2)}^2 \leq \nu, \text{ from which it follows that } \|C_\varphi\|_{\ell_2(\alpha^2)}^2 \leq \nu.
\]

In other words,

\[
\|C_\varphi(f)\|_{\ell_2(\alpha^2)}^2 \leq \nu^2 \|f\|_{\ell_2(\alpha^2)}^2.
\]

For all \( f \) in \( L_2^2(\alpha^2) \). Let \( n \) be any natural number and \( c_1, \ldots, c_n \) be complex numbers, and take

\[
 f = \sum_{j=1}^{n} c_j k_{\lambda_j}^{(\alpha+1)^2-1}.
\]

If we substitute this function into inequality (2), remembering that \( C_\varphi^*(k_{\lambda}^{(\alpha+1)^2-1}) = k_{\varphi(\lambda)}^{(\alpha+1)^2-1} \), we obtain

\[
\left\| \sum_{j=1}^{n} c_j k_{\lambda_j}^{(\alpha+1)^2-1} \right\|_{\ell_2(\alpha^2)}^2 \leq \nu^2 \left\| \sum_{j=1}^{n} c_j k_{\lambda_j}^{(\alpha+1)^2-1} \right\|_{\ell_2(\alpha^2)}^2.
\]

from which it follows that

\[
\sum_{j=1}^{n} |c_j|^2 \left\| k_{\lambda_j}^{(\alpha+1)^2-1} \right\|_{\ell_2(\alpha^2)}^2 \leq \nu^2 \sum_{j=1}^{n} |c_j|^2 \left\| k_{\lambda_j}^{(\alpha+1)^2-1} \right\|_{\ell_2(\alpha^2)}^2.
\]

and thus

\[
\sum_{j=1}^{n} |c_j|^2 \left( \frac{\nu^2}{(1-|\lambda_j|^2)^{(\alpha+1)^2+1}} - \frac{1}{(1-|\varphi(\lambda_j)|)^{(\alpha+1)^2+1}} \right) \geq 0.
\]

Inequality (3) is exactly the statement that \( M(\nu, n, \alpha(\alpha + 2)) \) is positive semidefinite.

For the converse, assume that \( M(\nu, n, \alpha(\alpha + 2)) \) is positive semidefinite for all natural numbers \( n \). Hence inequality (3) holds for all \( n \), which in turn implies that

\[
\left\| \sum_{j=1}^{n} c_j k_{\lambda_j}^{(\alpha+1)^2-1} \right\|_{\ell_2(\alpha^2)}^2 \leq \nu^2 \left\| \sum_{j=1}^{n} c_j k_{\lambda_j}^{(\alpha+1)^2-1} \right\|_{\ell_2(\alpha^2)}^2.
\]
For any $n$ and any complex constants $c_1, \ldots, c_n$. Now let $f$ be an arbitrary element of $L^2_{(\alpha+1)2^{-1}}$. Since $\Lambda$ is a set of uniqueness, the span of $\left\{K_{\lambda_j}^{(\alpha+1)2^{-1}}\right\}_{n=1}^{\infty}$ is dense in $L^2_{(\alpha+1)2^{-1}}$. Hence there exists a sequence $\left\{f_m\right\}_{m=1}^{\infty}$ that converges to $f$ in norm, where each $f_m$ is a finite linear combination of these kernel functions. The inequality of (4) implies that

$$\left\|C_{\rho}(f_m)\right\|_{L^2_{(\alpha+1)2^{-1}}}^2 \leq \nu^2 \left\|f_m\right\|_{L^2_{(\alpha+1)2^{-1}}}^2$$

for all $m$. Letting $m$ go to infinity, we see that

$$\left\|C_{\rho}(f)\right\|_{L^2_{(\alpha+1)2^{-1}}}^2 \leq \nu^2 \left\|f\right\|_{L^2_{(\alpha+1)2^{-1}}}^2$$

from which it follows (upon taking the supremum over all analytic functions $f$) that

$$\left\|C_{\rho}\right\|_{L^2_{(\alpha+1)2^{-1}}} \leq \nu$$.

Hence Proposition 1 states that $\left\|C_{\rho}\right\|_{L^2_{(\alpha+1)2^{-1}}} \leq \nu$ exactly when

$$k(\lambda, z) = \frac{\nu^2}{(1-\lambda z)^{(\alpha+1)2^{-1}}} - \frac{1}{\left(1-\overline{\varphi}(\lambda)\varphi(z)\right)^{(\alpha+1)2^{-1}}}$$

is a positive semidefinite kernel on the unit disk. □

2.2 Remark

If $f^r = \sum_{j=1}^{n} c_j k^{(\alpha+1)2^{-1}}_{\lambda_j}$ where $r = 1, 2, \ldots, n$. Proposition 1 implies that $f^r \rightarrow f^r$ uniformly in the norm. We can deduce that

$$\left\|C_{\rho}(f^r)\right\|_{L^2_{(\alpha+1)2^{-1}}}^2 \leq \nu^2 \sum_{j=1}^{n} \left\|k^{(\alpha+1)2^{-1}}_{\lambda_j}\right\|_{L^2_{(\alpha+1)2^{-1}}}^2$$

We need the following lemma which relating to positive semidefinite matrices.

2.3 Lemma

Let $n$ be a natural number and $\lambda_1, \ldots, \lambda_n$ be a finite collection of (not necessarily distinct) points in $D$. Any matrix of the form

$$M = \left[\frac{1}{(1-\lambda_j \lambda_i)^\rho}\right]_{j,i=1}^{n}$$

for any real number $\rho \geq 1$, must be positive semidefinite, and hence

$$M = \left[\frac{1}{(1-|\lambda|^2)^\rho}\right]_{j=1}^{n}$$

Proof. Let $\alpha = \sqrt{\rho - 1} - 1$ so that $\alpha \geq -1$. Taking $c = (c_1, c_2, \ldots, c_n) \in C^n$, we see that

$$\langle Mc, c \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i c_j}{(1-\lambda_j \lambda_i)^\alpha} = \left(\sum_{j=1}^{n} c_j k^{(\alpha+1)2^{-1}}_{\lambda_j}, \sum_{i=1}^{n} c_i k^{(\alpha+1)2^{-1}}_{\lambda_i}\right)_{L^2_{(\alpha+1)2^{-1}}} \geq 0,$$

from which our assertion follows and

$$\langle Mc, c \rangle = \left\|\sum_{j=1}^{n} c_j k^{(\alpha+1)2^{-1}}_{\lambda_j}\right\|_{L^2_{(\alpha+1)2^{-1}}}^2.$$
As a consequence of Lemma 2.3, we see that any matrix of the form
\[
\begin{bmatrix}
\frac{1}{1 - \varphi(\lambda_j)\varphi(\lambda_i)} \\
\frac{1}{1 - \varphi(\lambda_i)}
\end{bmatrix}_{i,j=1}^n,
\]
where \( \varphi \) is a self-map of \( D \), must also be positive semidefinite and hence
\[
\begin{bmatrix}
\frac{1}{1 - \varphi(\lambda_i)}^\alpha \\
\frac{1}{1 - \varphi(\lambda_i)}^\beta
\end{bmatrix}_{j=1}^n
\]
is required.

### 3. NORM INEQUALITIES

The proof of the major theorem relies heavily on the use of Schur products. Recall that, for any two \( n \times n \) matrices \( A = [a_{i,j}]_{i,j=1}^n \) and \( B = [b_{i,j}]_{i,j=1}^n \), the Schur (or Hadamard) product \( A \circ B \) is defined by the following rule
\[
A \circ B = [a_{i,j}b_{i,j}]_{i,j=1}^n.
\]
That is, the Schur product is obtained by entrywise multiplication. A proof of the following results appears in [12].

#### 3.1 Proposition

(Schur Product Theorem). If \( A \) and \( B \) are \( n \times n \) positive semidefinite matrices, then \( A \circ B \) is also positive semidefinite.

We are now in position to state the main result, a theorem that allows us to compare the norms of \( C_{\varphi} \) on certain weighted spaces.

#### 3.2 Theorem

Take \( \beta + 1 \geq (\alpha + 1)^2 \geq 0 \) and let \( \varphi \) be an analytic self-map of \( D \). Then
\[
\left\| C_{\varphi} \right\|_{L^\beta_{\alpha+1}} \leq \left\| C_{\varphi} \right\|_{L^\gamma_{\alpha+1}},
\]
(5)
Whenever the quantity \( \gamma = \frac{\beta + 2}{(\alpha + 1)^2 + 1} \) is an integer.

**Proof.** Assume that \( \gamma = \frac{\beta + 2}{(\alpha + 1)^2 + 1} \) is an integer. Fix a natural number \( n \) and let \( i, j \in \{1, 2, \ldots, n\} \). A difference of higher powers factorization shows that
\[
\left\| C_{\varphi} \right\|_{L^\gamma_{\alpha+1}} = \left( \frac{1}{1 - \lambda_j \lambda_i} \right)^{\beta + 2} \left( \frac{1}{1 - \varphi(\lambda_j)\varphi(\lambda_i)} \right)^{\beta + 2}
\]
\[
= \left( \frac{1}{1 - \lambda_j \lambda_i} \right)^{(\alpha + 1)^2 + 1} \left( \frac{1}{1 - \varphi(\lambda_j)\varphi(\lambda_i)} \right)^{(\alpha + 1)^2 + 1} \sum_{k=0}^{\gamma-1} \left( \frac{1}{1 - \lambda_j \lambda_i} \right)^{(\alpha + 1)^2 + 1} \left( \frac{1}{1 - \varphi(\lambda_j)\varphi(\lambda_i)} \right)^{(\alpha + 1)^2 + 1}
\]
then
(1−|λj|2)β+2 \left(1−|φ(λj)|^2\right)^β+2 \right) \\
\leq \sum_{\kappa=0}^{\infty} \left(\sum_{i=0}^{2^{\kappa}} \frac{1}{\left(1−|λj|\right)^{(α+1)\kappa+i+1}} \sum_{j=0}^{2^{\kappa}} \frac{1}{\left(1−|φ(λj)|\right)^{(α+1)\kappa+i+1}} \right) \left(1−|φ(λj)|^2\right)^β+2 \\
\left(1−|φ(λj)|^2\right)^β+2 \\
Since the preceding equation holds for all \( i \) and \( j \), we obtain the following matrix equation:

\[ M \left( \left\| C_{φ} \right\|_{(α+1)^2−1}, n, β \right) = M \left( \left\| C_{φ} \right\|_{(α+1)^2−1}, n, α(α+2) \right) \sum_{\kappa=0}^{\infty} \left( \sum_{i=0}^{2^{\kappa}} \frac{1}{\left(1−|λj|\right)^{(α+1)\kappa+i+1}} \sum_{j=0}^{2^{\kappa}} \frac{1}{\left(1−|φ(λj)|\right)^{(α+1)\kappa+i+1}} \right) \left(1−|φ(λj)|^2\right)^β+2 \\
\text{Then we can show the result when } i = j .

The matrix \( M \left( \left\| C_{φ} \right\|_{(α+1)^2−1}, n, α(α+2) \right) \) is positive semidefinite by Proposition 2.1. Lemma 2.3, together with the Schur Product Theorem, dictates that every term in the matrix sum on the right-hand side of (6) is positive semidefinite, so the sum itself is positive semidefinite. Therefore the Schur Product Theorem shows that

\[ M \left( \left\| C_{φ} \right\|_{(α+1)^2−1}, n, β \right) \] must also be positive semidefinite.

Since this assertion holds for every natural number \( n \), Proposition 2.1. shows that \( \left\| C_{φ} \right\|_{(α+1)^2−1} \leq \left\| C_{φ} \right\|_{(α+1)^2−1} \).

Taking \( α = 0 \), \( α = -1 \) and \( α = -2 \), we obtain (see [11]) the following corollaries.

### 3.3 Corollary

Let \( φ \) be an analytic self-map of \( D \). Then

\[ \left\| C_{φ} \right\|_{(α+1)^2−1} \leq \left\| C_{φ} \right\|_{(α+1)^2−1} \]

whenever \( β \) is a non-negative integer. In particular,

\[ \left\| C_{φ} \right\|_{(α+1)^2−1} \leq \left\| C_{φ} \right\|_{(α+1)^2−1} \]

### 3.4 Corollary

Let \( φ \) be an analytic self-map of \( D \). Then

\[ \left\| C_{φ} \right\|_{(α+1)^2−1} \leq \left\| C_{φ} \right\|_{(α+1)^2−1} \]

whenever \( β \) is a positive even integer.

### 3.5 Theorem

Take \( β+1\geq(α+1)^2 \geq 0 \) and let \( φ \) be an analytic self-map of \( D \). Suppose that \( γ = \frac{β+2}{(α+1)^2+1} \) is an integer. If \( C_{φ} \) is cosubnormal on \( L^2_{(α+1)^2−1} \), then it is also cosubnormal on \( L^2_{β} \).
Cowen only stated this result for $\alpha = -1$ [13], but an identical argument works for $\alpha > -1$. The proof makes use of the Schur Product Theorem in a similar fashion to that of Theorem 3.2. (See [11], [14], and [15]).

REFERENCES


